# A note on robot kinematics

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# 1 Introduction

This note presents robot kinematics, which is the geometric description of robot motion. The presentation is based on the material in robotic textbooks like [16, 17]. In addition, there is material on screw theory [9], which is used in the robotic textbook [11]. The note provides the reader with additional background and details on these topics. In particular, the note gives insight on both coordinate-free descriptions of vectors and dyadics, and on the corresponding coordinate formulations using column vectors and matrices. The description of rotations is given much attention, and also the description of the position and orientation of a rigid bodies is treated in detail.

# 2 Vectors

## 2.1 Vector description

Forces, torques, velocities and accelerations are well-known entities that can be described by vectors. A vector $\vec{u}$ can be described by its magnitude $|\vec{u}|$ and its direction.

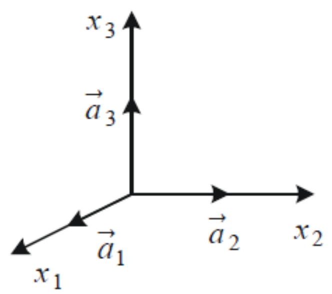



Figure 1: The coordinate frame $a$.

Note that this description of a vector does not rely on the definition of any coordinate frame, and the vector is said to be coordinate-free. Let a Cartesian coordinate frame be defined by three orthogonal unit vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ that are unit vectors along the $x, y, z$ axes of the frame (Figure 1). Then the vector $\vec{u}$ can be expressed as a linear combination of the orthogonal unit vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ by

$$\vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3 \tag{1}$$

where

$$u_i = \vec{u} \cdot \vec{e}_i, \quad i \in \{1, 2, 3\} \tag{2}$$

are the unique *components* or *coordinates* of $\vec{u}$ in $a$.

A related description of the vector is given in terms of the *coordinate vector* which is a column vector

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{3}$$

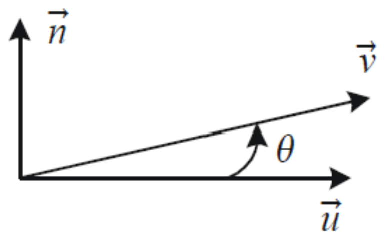



Figure 2: The vectors $\vec{u}$, $\vec{v}$ and $\vec{n}$.

### 2.2 The scalar product

The *scalar product* between two vectors $\vec{u}$ and $\vec{v}$ is given in the coordinate-free description by

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta \tag{4}$$

where $\theta$ is the angle between the two vectors (Figure 2).

In coordinate form with reference to the frame $a$ we may then represent the vectors $\vec{u}$ and $\vec{v}$ by their coordinate vectors

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{5}$$

where $u_i = \vec{u} \cdot \vec{e}_i$ and $v_i = \vec{v} \cdot \vec{e}_i$. The scalar product in terms of coordinate vectors is

$$\begin{aligned} \vec{u} \cdot \vec{v} &= (u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3) \cdot (v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3) \\ &= u_1v_1 + u_2v_2 + u_3v_3 \\ &= \boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} \end{aligned}$$

where it is used that $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$ which is equal to unity when $i = j$ and zero otherwise.

The scalar product can be written in the three alternative forms

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{3} u_i v_i = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} \tag{6}$$

### 2.3 The vector cross product

The *vector cross product* is given in the coordinate-free form by

$$\vec{u} \times \vec{v} = \vec{n}|\vec{u}||\vec{v}| \sin\theta \tag{7}$$

where $0 \leq \theta \leq \pi$ and $\vec{n}$ is a unit vector that is orthogonal to both $\vec{u}$ and $\vec{v}$ and defined so that $(\vec{u}, \vec{v}, \vec{n})$ forms a right-hand system (Figure 2).

Using the components in the Cartesian frame with unit vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ the vector cross product can be evaluated from

$$\vec{w} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{8}$$

which gives

$$\vec{w} = (u_2v_3 - u_3v_2)\vec{e}_1 + (u_3v_1 - u_1v_3)\vec{e}_2 + (u_1v_2 - u_2v_1)\vec{e}_3 \tag{9}$$

In coordinate vector notation we introduce the *skew-symmetric form* of the coordinate vector $\boldsymbol{u}$ defined by

$$\boldsymbol{u}^{\times} := \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \tag{10}$$

Then the vector cross product can be written in coordinate vector form as

$$\boldsymbol{w} = \boldsymbol{u}^{\times}\boldsymbol{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

It is concluded that the vector cross product has the following equivalent representations:

$$\vec{w} = \vec{u} \times \vec{v} \;\Leftrightarrow\; \boldsymbol{w} = \boldsymbol{u}^{\times}\boldsymbol{v} \tag{11}$$

It is noted that a set of orthogonal unit vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ satisfies

$$\vec{e}_1 \times \vec{e}_2 = \vec{e}_3, \quad \vec{e}_1 \times \vec{e}_3 = -\vec{e}_2 \tag{12}$$

$$\vec{e}_2 \times \vec{e}_3 = \vec{e}_1, \quad \vec{e}_2 \times \vec{e}_1 = -\vec{e}_3 \tag{13}$$

$$\vec{e}_3 \times \vec{e}_1 = \vec{e}_2, \quad \vec{e}_3 \times \vec{e}_2 = -\vec{e}_1 \tag{14}$$

### 2.4 The triple scalar product

Let $\vec{a}, \vec{b}, \vec{c}$ be three arbitrary vectors. Then the triple scalar product satisfies product is

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{n} \cdot \vec{c} = |\vec{n}||\vec{c}| \cos\theta \tag{15}$$

where $\vec{n} = \vec{a} \times \vec{b}$, $|\vec{n}|$ is equal to the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$, and $\theta$ is the angle between $\vec{n}$ and $\vec{c}$. The triple scalar product is equal to the signed volume of the parallelepiped spanned by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

From the the geometric argument that the triple scalar product is the volume spanned by the three vectors, it follows that

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{c} \times \vec{a}) \cdot \vec{b} = (\vec{b} \times \vec{c}) \cdot \vec{a} \tag{16}$$

It is interesting to note that

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \left( \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{e}_1 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{e}_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{e}_3 \right) \cdot (c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3) \tag{17}$$

which means that the volume spanned by the three vectors can be expressed by a determinant according to

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{18}$$

The rows of the determinant can be interchanged in a cyclical order, that is,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \tag{19}$$

which confirms the result (16).

### 2.5 The triple vector product

Let $\vec{a}, \vec{b}, \vec{c}$ be three arbitrary vectors. Then the triple vector product satisfies

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b}\vec{c} \tag{20}$$

This can be shown by calculation of the components on both sides. Let $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ be the coordinate representations of $\vec{a}$, $\vec{b}$ and $\vec{c}$ in some coordinate frame. The coordinate form of the triple vector product is then

$$\boldsymbol{a}^{\times}\boldsymbol{b}^{\times}\boldsymbol{c} = \boldsymbol{b}\boldsymbol{a}^{\mathrm{T}}\boldsymbol{c} - \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}\boldsymbol{c} = (\boldsymbol{b}\boldsymbol{a}^{\mathrm{T}} - \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}\boldsymbol{I})\boldsymbol{c} \tag{21}$$

The vector $\boldsymbol{c}$ is arbitrary, and it follows that

$$\boldsymbol{a}^{\times}\boldsymbol{b}^{\times} = \boldsymbol{b}\boldsymbol{a}^{\mathrm{T}} - \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}\boldsymbol{I} \tag{22}$$

In particular we note that

$$\boldsymbol{a}^{\times}\boldsymbol{a}^{\times} = \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}} - \boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}\boldsymbol{I}. \tag{23}$$

For a unit vector $\boldsymbol{k}$ this gives

$$\boldsymbol{k}^{\times}\boldsymbol{k}^{\times} = \boldsymbol{k}\boldsymbol{k}^{\mathrm{T}} - \boldsymbol{I} \tag{24}$$

since $\boldsymbol{k}^{\mathrm{T}}\boldsymbol{k} = 1$.

The triple vector product satisfies the Jacobi identity

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0} \tag{25}$$

The coordinate form of this is

$$\boldsymbol{a}^{\times}\boldsymbol{b}^{\times}\boldsymbol{c} + \boldsymbol{b}^{\times}\boldsymbol{c}^{\times}\boldsymbol{a} + \boldsymbol{c}^{\times}\boldsymbol{a}^{\times}\boldsymbol{b} = \boldsymbol{0} \tag{26}$$

The terms in the Jacobi identity can be rearranged to give

$$(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c}) - \vec{b} \times (\vec{a} \times \vec{c}) \tag{27}$$

the coordinate form of this is

$$(\boldsymbol{a}^{\times}\boldsymbol{b})^{\times}\boldsymbol{c} = \boldsymbol{a}^{\times}\boldsymbol{b}^{\times}\boldsymbol{c} - \boldsymbol{b}^{\times}\boldsymbol{a}^{\times}\boldsymbol{c} \tag{28}$$

Since $\boldsymbol{c}$ is arbitrary, this gives

$$(\boldsymbol{a}^{\times}\boldsymbol{b})^{\times} = \boldsymbol{a}^{\times}\boldsymbol{b}^{\times} - \boldsymbol{b}^{\times}\boldsymbol{a}^{\times} \tag{29}$$

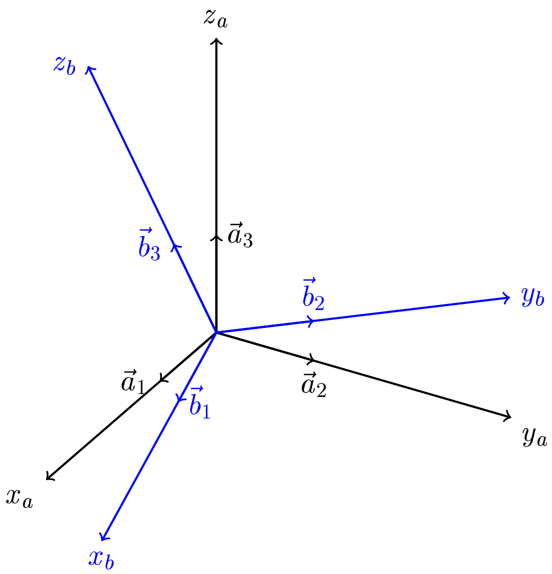



Figure 3: The coordinate frames $a$ and $b$.

# 3 The rotation matrix

## 3.1 Coordinate transformations for vectors

It was shown that a vector can be described in terms of its component in a coordinate frame $a$ with orthogonal unit vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$. Dynamic models for use in robotics, car dynamics, aerospace, marine systems, and navigation typically involve several Cartesian frames, so that a vector may have to be described in more than one frame. To investigate this we introduce a second coordinate frame $b$ with orthogonal unit vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3$ along the axes. A vector $\vec{v}$ may then be represented with respect to the systems $a$ and $b$ using the notation

$$\vec{v} = \sum_{i=1}^{3} v_i^a \vec{a}_i \quad \text{and} \quad \vec{v} = \sum_{i=1}^{3} v_i^b \vec{b}_i \tag{30}$$

where

$$v_i^a = \vec{v} \cdot \vec{a}_i \tag{31}$$

are the coordinates of $\vec{v}$ in $a$, and

$$v_i^b = \vec{v} \cdot \vec{b}_i \tag{32}$$

are the coordinates of $\vec{v}$ in $b$. To distinguish the column vectors of coordinates in frame $a$ from the column vector of coordinates in frame $b$ we write

$$\boldsymbol{v}^a = \begin{bmatrix} v_1^a \\ v_2^a \\ v_3^a \end{bmatrix} \quad \text{and} \quad \boldsymbol{v}^b = \begin{bmatrix} v_1^b \\ v_2^b \\ v_3^b \end{bmatrix} \tag{33}$$

where superscript $a$ denotes that the vector is given by the the coordinates in $a$, and the superscript $b$ denotes that the vector is given by the coordinates in $b$.

To find the relation between the coordinate vectors $\boldsymbol{v}^a$ and $\boldsymbol{v}^b$ in frames $a$ and $b$ the following calculation is used:

$$\begin{aligned} v_i^a &= \vec{v} \cdot \vec{a}_i = (v_1^b \vec{b}_1 + v_2^b \vec{b}_2 + v_3^b \vec{b}_3) \cdot \vec{a}_i \\ &= \sum_{j=1}^{3} v_j^b \left( \vec{a}_i \cdot \vec{b}_j \right) \end{aligned} \tag{34}$$

This leads to the following result: The coordinate transformation from frame $b$ to frame $a$ is given by

$$\boldsymbol{v}^a = \boldsymbol{R}_b^a \boldsymbol{v}^b \tag{35}$$

where

$$\boldsymbol{R}_b^a = \{\vec{a}_i \cdot \vec{b}_j\} \tag{36}$$

is called the *rotation matrix* from $a$ to $b$. The elements $r_{ij} = \vec{a}_i \cdot \vec{b}_j$ of the rotation matrix $\boldsymbol{R}_b^a$ are called the *direction cosines.*

It is seen that the rotation matrix from $a$ to $b$ transforms a vector given in the coordinates of frame $b$ to the same vector given in the coordinates of frame $a$. Because of this the matrix may also be called the *coordinate transformation matrix* from $b$ to $a$.

### 3.2 Properties of the rotation matrix

The rotation matrix has a number of useful properties that will be discussed in this section. First it is noted that the rotation matrix from $b$ to $a$ can be found in the same way as the rotation matrix from $a$ to $b$ by simply interchanging $a$ and $b$ in the expressions. This gives

$$\boldsymbol{R}_a^b = \{\vec{b}_i \cdot \vec{a}_j\} \tag{37}$$

For all $\boldsymbol{v}^b$ we have

$$\boldsymbol{v}^b = \boldsymbol{R}_a^b \boldsymbol{v}^a = \boldsymbol{R}_a^b \boldsymbol{R}_b^a \boldsymbol{v}^b \tag{38}$$

This implies that

$$\boldsymbol{R}_a^b \boldsymbol{R}_b^a = \boldsymbol{I}, \tag{39}$$

and it follows that

$$\boldsymbol{R}_a^b = (\boldsymbol{R}_b^a)^{-1} \tag{40}$$

A comparison of the elements in the matrices in (36) and (37) leads to the conclusion that $\boldsymbol{R}_a^b = (\boldsymbol{R}_b^a)^{\mathrm{T}}$. This shows that the rotation matrix is orthogonal and satisfies

$$\boldsymbol{R}_a^b = (\boldsymbol{R}_b^a)^{-1} = (\boldsymbol{R}_b^a)^{\mathrm{T}} \tag{41}$$

Consider a vector $\vec{p}$ with coordinate vector $\boldsymbol{p}^a$ in frame $a$. Define the vector $\vec{q}$ defined by its coordinate vector

$$\boldsymbol{q}^a = \boldsymbol{R}_b^a \boldsymbol{p}^a \tag{42}$$

Note that the vector $\vec{q}$ is defined by the vector $\vec{p}$ and the rotation matrix $\boldsymbol{R}_b^a$. The coordinate vector $\boldsymbol{q}^b$ in $b$ is according to the usual coordinate transformation rule

$$\boldsymbol{q}^b = \boldsymbol{R}_a^b \boldsymbol{q}^a = \boldsymbol{R}_a^b \boldsymbol{R}_b^a \boldsymbol{p}^a = \boldsymbol{p}^a \tag{43}$$

which means that the coordinates of $\vec{q}$ in $b$ are equal to the coordinates of $\vec{p}$ in $a$. This shows that the rotation matrix from $a$ to $b$ rotates the vector $\vec{p}$ to the vector $\vec{q}$ so that $\boldsymbol{q}^b = \boldsymbol{p}^a$.

From this it follows that

$$\boldsymbol{b}_i^a = \boldsymbol{R}_b^a \boldsymbol{a}_i^a \tag{44}$$

which means that the rotation matrix $\boldsymbol{R}_b^a$ rotates the orthogonal unit vector $\vec{a}_i$ in $a$ to the orthogonal unit vectors $\vec{b}_i$ in $b$. This is seen from

$$\boldsymbol{a}_1^a = \boldsymbol{b}_1^b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{a}_2^a = \boldsymbol{b}_2^b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{a}_3^a = \boldsymbol{b}_3^b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{45}$$

It follows that the columns of the rotation matrix are the coordinate vectors $\boldsymbol{b}_i^a$ of $\vec{b}_i$ in frame $a$, that is

$$\boldsymbol{R}_b^a = \begin{bmatrix} \boldsymbol{b}_1^a & \boldsymbol{b}_2^a & \boldsymbol{b}_3^a \end{bmatrix} \tag{46}$$

Note that the rotation matrix $\boldsymbol{R}_b^a$ from $a$ to $b$ has two interpretations:

1. Consider two frames $a$ and $b$ with origin in the same point. Let the vector $\vec{v}$ have coordinate vector $\boldsymbol{v}^b$ in $b$ and coordinate vector $\boldsymbol{v}^a$ in $a$. Then the rotation matrix $\boldsymbol{R}_b^a$ transforms the coordinate vector in $b$ to the coordinate vector in $a$ according to

$$\boldsymbol{v}^a = \boldsymbol{R}_b^a \boldsymbol{v}^b \tag{47}$$

In this equation $\boldsymbol{R}_b^a$ acts as a coordinate transformation matrix.

2. The vector $\vec{p}$ with coordinate vector $\boldsymbol{p}^a$ in $a$ is rotated to the vector $\vec{q}$ with coordinate vector $\boldsymbol{q}^b = \boldsymbol{p}^a$ by

$$\boldsymbol{q}^a = \boldsymbol{R}_b^a \boldsymbol{p}^a \tag{48}$$

In this equation $\boldsymbol{R}_b^a$ acts as a rotation matrix.

The determinant of the rotation matrix $\boldsymbol{R}_b^a$ is found by direct calculation to be

$$\begin{aligned} \det \boldsymbol{R}_b^a &= r_{11}(r_{22}r_{33} - r_{32}r_{23}) + r_{21}(r_{32}r_{13} - r_{12}r_{33}) + r_{31}(r_{12}r_{23} - r_{22}r_{13}) \\ &= (\boldsymbol{b}_1^a)^{\mathrm{T}} \left[ (\boldsymbol{b}_2^a)^{\times} \boldsymbol{b}_3^a \right] = (\boldsymbol{b}_1^a)^{\mathrm{T}} \boldsymbol{b}_1^a = 1 \end{aligned}$$

where it is used that $(\boldsymbol{b}_2^a)^{\times} \boldsymbol{b}_3^a = \boldsymbol{b}_1^a$, and that $\boldsymbol{b}_1^a$ is a unit vector. This means that the rotation matrix has a determinant equal to unity, that is

$$\det \boldsymbol{R}_b^a = 1 \tag{49}$$

Finally, the set $SO(3)$ is defined. We have established that the rotation matrix is orthogonal and has a determinant equal to unity. The set of all matrices that are orthogonal and with a determinant equal to unity is denoted by $SO(3)$, that is,

$$SO(3) = \{\boldsymbol{R} | \boldsymbol{R} \in \mathbb{R}^{3\times 3}, \quad \boldsymbol{R}^{\mathrm{T}}\boldsymbol{R} = \boldsymbol{I} \quad \text{and} \quad \det \boldsymbol{R} = 1\} \tag{50}$$

Here $\mathbb{R}^{3\times 3}$ is the set of all $3 \times 3$ matrices with real elements. A matrix $\boldsymbol{R}$ is a rotation matrix if and only if it is an element of the set $SO(3)$.

### 3.3 Composite rotations

The rotation from frame $a$ to a frame $c$ may be described as a *composite rotation* made up by a rotation from $a$ to $b$, and then a rotation from $b$ to $c$. The transformation of $\boldsymbol{v}^c$ to $b$ and to $a$ is given by

$$\begin{aligned} \boldsymbol{v}^b &= \boldsymbol{R}_c^b \boldsymbol{v}^c \\ \boldsymbol{v}^a &= \boldsymbol{R}_c^a \boldsymbol{v}^c \end{aligned}$$

Combining these two equations we get

$$\boldsymbol{v}^a = \boldsymbol{R}_b^a \boldsymbol{v}^b = \boldsymbol{R}_b^a \boldsymbol{R}_c^b \boldsymbol{v}^c$$

This shows that:

The rotation matrix of a composite rotation is the product of the rotation matrices:

$$\boldsymbol{R}_c^a = \boldsymbol{R}_b^a \boldsymbol{R}_c^b$$

This shows that the rotation matrix for the composite rotation $\boldsymbol{R}_c^a$ is simply the product of the rotation matrices $\boldsymbol{R}_b^a$ from $a$ to $b$ and $\boldsymbol{R}_c^b$ from $b$ to $c$. It is straightforward to extend this result to the composite rotation of three or more rotations. In the case of three rotations we have

$$\boldsymbol{R}_d^a = \boldsymbol{R}_b^a \boldsymbol{R}_c^b \boldsymbol{R}_d^c \tag{51}$$

### 3.4 Simple rotations

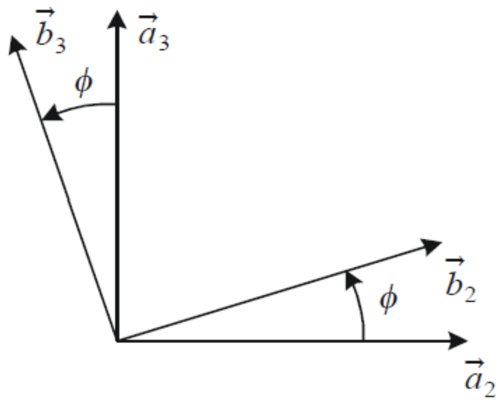



Figure 4: A rotation by an angle $\theta$ around $\vec{a_1}$.

A rotation about a fixed axis is called a *simple rotation*. We will here derive the rotation matrices corresponding to simple rotations about the $x$, $y$ and $z$ axes. Consider a rotation by an angle $\phi$ about the $x_a$ axis from a frame $a$ to a frame $b$. The resulting rotation matrix is denoted $\boldsymbol{R}_x(\phi)$. In the same way we define $\boldsymbol{R}_y(\theta)$ to be the rotation by an angle $\theta$ about the $y$ axis, and $\boldsymbol{R}_z(\psi)$ to be the rotation by an angel $\psi$ about the $z$ axis.

For the rotation $\boldsymbol{R}_x(\phi)$ we see from Figure 4 that $\vec{a}_1 = \vec{b}_1$, so that $\vec{a}_1 \cdot \vec{b}_1 = 1$, while

$$\begin{aligned} \vec{a}_1 \cdot \vec{b}_2 &= \vec{a}_1 \cdot \vec{b}_3 = \vec{a}_2 \cdot \vec{b}_1 = \vec{a}_3 \cdot \vec{b}_1 = 0 & (52) \\ \vec{a}_2 \cdot \vec{b}_2 &= \cos\phi, \quad \vec{a}_3 \cdot \vec{b}_3 = \cos\phi & (53) \\ \vec{a}_3 \cdot \vec{b}_2 &= \sin\phi, \quad \vec{a}_2 \cdot \vec{b}_3 = -\sin\phi & (54) \end{aligned}$$

In the same way we can find the elements of the matrices $\boldsymbol{R}_y(\theta)$ and $\boldsymbol{R}_z(\psi)$. This results in

$$\boldsymbol{R}_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \tag{55}$$

$$\boldsymbol{R}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{56}$$

$$\boldsymbol{R}_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{57}$$

### 3.5 Angle-axis parameters

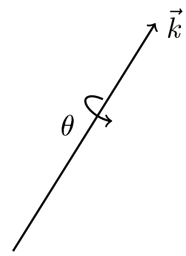



Figure 5: Angle axis parameters $\theta$ and $\vec{k}$ of a rotation where $\vec{k}$ is a unit vector, and $\theta$ is the angle of the rotation about $\vec{k}$.

A rotation matrix $\boldsymbol{R}_b^a$ is orthogonal with determinant equal to unity. It can be shown [1] that this implies that one of the eigenvalues to the matrix is equal to one, and it follows that a rotation matrix ha a eigenvector $\boldsymbol{k}$ which satisfies

$$\boldsymbol{R}_b^a \boldsymbol{k} = \boldsymbol{k} \tag{58}$$

This algebraic result can be given a geometric interpretation which is the basis for the *angle-axis parametrization* of the rotation matrix $\boldsymbol{R}_b^a$. The geometric interpretation that will be used is that the eigenvector $\boldsymbol{k}$ is the coordinate vector of a unit vector $\vec{k}$, where $\vec{k}$ is defined by its coordinate vector

$$\boldsymbol{k}^a = \boldsymbol{k} \tag{59}$$

in frame $a$. The transformation rule

$$\boldsymbol{k}^a = \boldsymbol{R}_b^a \boldsymbol{k}^b \tag{60}$$

then implies that

$$\boldsymbol{k}^a = \boldsymbol{k}^b = \boldsymbol{k} \tag{61}$$

which means that $\vec{k}$ has the same coordinates in frames $a$ and $b$. It is therefore possible to describe the rotation from $a$ to $b$ as a simple rotation by an angle $\theta$ about the vector $\vec{k}$ which is fixed in both $a$ and $b$. On background of this $(\theta, \vec{k})$ is called the *angle-axis parametrization* of the rotation matrix $\boldsymbol{R}_b^a$. Note that this gives four parameters and one constraint equation, namely the angle $\theta$ plus the three coordinates of the unit vector $\vec{k}$, and the constraint equation $\vec{k} \cdot \vec{k} = 1$.

### 3.6 Derivation of the rotation matrix of a general simple rotation

In this section the rotation matrix $\boldsymbol{R}$ corresponding to a simple rotation $\theta$ about a fixed unit vector $\vec{k}$ will be presented. The result is called Rodrigues equation and is attributed to Olinde Rodrigues [13]. The derivation will be based on a coordinate-free description. This has the advantage that the result can be represented in an arbitrary coordinate frame, and it is straightforward to transform the result to another coordinate frame. The derivation is based on the active form of the rotation matrix where a vector $\boldsymbol{p}$ is rotated to a vector $\boldsymbol{q}$ according to $\boldsymbol{q} = \boldsymbol{R}\boldsymbol{p}$. The coordinate vector $\boldsymbol{p}$ is a column vector of the coordinates of the vector $\vec{p}$, where $\vec{p}$ is given in terms of its magnitude and direction. In the same way, the coordinate vector $\boldsymbol{q}$ is a column vector of the coordinates of the vector $\vec{q}$, where $\vec{q}$ is given in terms of its magnitude and direction.

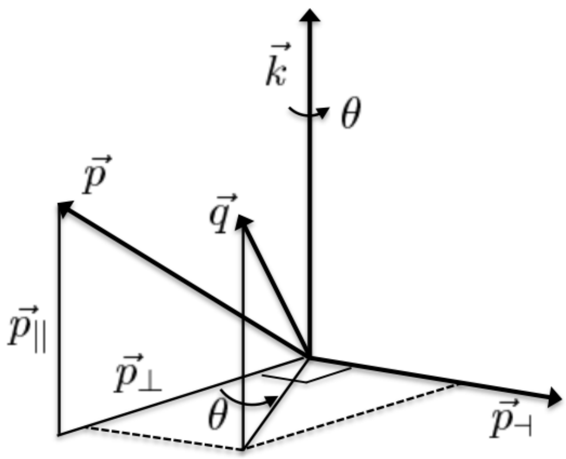



Figure 6: Rotation of a vector $\vec{p}$ to a vector $\vec{q}$ by an angle $\theta$ about $\vec{k}$.

Consider the vector $\vec{p}$, and suppose that it is rotated to a vector $\vec{q}$ by an angle $\theta$ about a unit vector $\vec{k}$. The vector $\vec{p}$ is decomposed into two components as

$$\vec{p} = \vec{p}_{||} + \vec{p}_{\perp} \tag{62}$$

where the component $\vec{p}_{||}$ is along $\vec{k}$, and the component $\vec{p}_{\perp}$ is perpendicular to $\vec{k}$. The component along $\vec{k}$ is

$$\vec{p}_{||} = \vec{k}(\vec{k} \cdot \vec{p})$$

while the component perpendicular to $\vec{k}$ is

$$\vec{p}_\perp = \vec{p} - \vec{p}_{||} = \vec{p} - \vec{k}(\vec{k} \cdot \vec{p})$$

Next, consider the vector

$$\vec{p}_\dashv = \vec{k} \times \vec{p}$$

which is perpendicular to both $\vec{p}$ and $\vec{k}$. Then the following calculation gives an interesting and useful result:

$$\vec{p}_\dashv \times \vec{k} = (\vec{k} \times \vec{p}) \times \vec{k} = \vec{p}(\vec{k} \cdot \vec{k}) - \vec{k}(\vec{k} \cdot \vec{p}) = \vec{p} - \vec{k}(\vec{k} \cdot \vec{p}) = \vec{p}_\perp$$

This implies that $|\vec{p}_\perp| = |\vec{p}_\dashv|$, which follows as $\vec{p}_\dashv$ is perpendicular to $\vec{k}$. This means that the two vectors $\vec{p}_\perp$ and $\vec{p}_\dashv$, which are both in the plane perpendicular to $\vec{k}$, will have equal magnitude, and will be orthogonal to each other, so that $\vec{p}_\perp$, $\vec{p}_\dashv$ and $\vec{k}$ will be orthogonal vectors that form a right-hand system.

After these preliminaries, the expression for the rotation matrix can be derived. In the rotation of $\vec{p} = \vec{p}_{||} + \vec{p}_\perp$ by an angle $\theta$ about $\vec{k}$, the component $\vec{p}_{||}$ along $\vec{k}$ will not be changed, while the projection $\vec{p}_\perp$ in the plane will be rotated by an angle $\theta$ to $\cos\theta\vec{p}_\perp + \sin\theta\vec{p}_\dashv$. The result is

$$\vec{q} = \vec{p}_{||} + \cos\theta\vec{p}_\perp + \sin\theta\vec{p}_\dashv \tag{63}$$

Insertion of the expressions for $\vec{p}_{||}$, $\vec{p}_\perp$ and $\vec{p}_\dashv$ gives

$$\vec{q} = \vec{k}(\vec{k} \cdot \vec{p}) - \cos\theta\vec{k} \times (\vec{k} \times \vec{p}) + \sin\theta\vec{k} \times \vec{p} \tag{64}$$

Insertion of the general expression $\vec{k} \times (\vec{k} \times \vec{p}) = \vec{k}(\vec{k} \cdot \vec{p}) - \vec{p}$ leads to Rodrigues equation

$$\vec{q} = \vec{p} + \sin\theta\vec{k} \times \vec{p} + (1 - \cos\theta)\vec{k} \times (\vec{k} \times \vec{p}) \tag{65}$$

which is the expression for a vector $\vec{q}$ that rotated by an angle $\theta$ about there unit vector $\vec{k}$. Note that this derivation was done without the introduction of any coordinate frame, and because of this the result can be said to be based on a coordinate-free description.

The coordinate form of the result is obtained simply by substituting the coordinate vectors $\boldsymbol{q}$, $\boldsymbol{p}$ and $\boldsymbol{k}$, where all three coordinate vectors must be in the same coordinate frame. The result is

$$\boldsymbol{q} = \boldsymbol{p} + \sin\theta\boldsymbol{k}^\times\boldsymbol{p} + (1 - \cos\theta)\boldsymbol{k}^\times\boldsymbol{k}^\times\boldsymbol{p} \tag{66}$$

Here $\boldsymbol{p}$ can be factored out, which gives

$$\boldsymbol{q} = \left(\boldsymbol{I} + \sin\theta\boldsymbol{k}^\times + (1 - \cos\theta)\boldsymbol{k}^\times\boldsymbol{k}^\times\right)\boldsymbol{p} \tag{67}$$

Then, since $\boldsymbol{q} = \boldsymbol{R}\boldsymbol{p}$, it follows that the rotation matrix is given by

$$\boldsymbol{R} = \boldsymbol{I} + \sin\theta\boldsymbol{k}^\times + (1 - \cos\theta)\boldsymbol{k}^\times\boldsymbol{k}^\times \tag{68}$$

It follows from $\boldsymbol{k}^\times\boldsymbol{k} = \boldsymbol{0}$ that

$$\boldsymbol{R}\boldsymbol{k} = \boldsymbol{k} \tag{69}$$

This means that the vector $\boldsymbol{k}$ is not rotated by the rotation matrix, which is reasonable, as the vector is along the rotation axis.

**Example 1**

Suppose that the coordinate frame is selected so that $\vec{k}$ is along the $z$ axis. Then $\boldsymbol{k}_z = [0, 0, 1]^{\mathrm{T}}$, and

$$\boldsymbol{R}_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (1 - \cos\theta) \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{70}$$

which gives the familiar result

$$\boldsymbol{R}_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{71}$$

of a rotation about the $z$ axis. □

**Example 2**

Suppose that the coordinate frame is selected so that $\vec{k}$ is along the $x$ axis. Then $\boldsymbol{k}_x = [1, 0, 0]^{\mathrm{T}}$, and

$$\boldsymbol{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + (1 - \cos\theta) \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{72}$$

This gives

$$\boldsymbol{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{73}$$

which is a is known as a rotation about the $x$ axis. In the same way the rotation matrix

$$\boldsymbol{R}_y = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{74}$$

is found is the coordinate frame is selected so that $\vec{k}$ is along the $y$ axis. □

**Example 3** An alternative expression is obtained by using $\boldsymbol{k}^{\times}\boldsymbol{k}^{\times} = \boldsymbol{k}\boldsymbol{k}^{\mathrm{T}} - \boldsymbol{I}$. Then the rotation matrix can be written

$$\boldsymbol{R} = \cos\theta \boldsymbol{I} + \sin\theta \boldsymbol{k}^{\times} + (1 - \cos\theta)\boldsymbol{k}\boldsymbol{k}^{\mathrm{T}} \tag{75}$$

□

### 3.7 Transformation of the rotation matrix

Consider the rotation

$$\vec{p} = \vec{q} + \sin\theta \vec{k} \times \vec{q} + (1 - \cos\theta)\vec{k} \times (\vec{k} \times \vec{q}) \tag{76}$$

of the vector $\vec{q}$ to the vector $\vec{p}$ defined by the rotation by an angle $\theta$ about $\vec{k}$. Let the rotation be described in the $a$ frame. Then the rotation is written $\boldsymbol{p}^a = \boldsymbol{R}^a \boldsymbol{q}^a$. Then, since $\boldsymbol{p}^a$ and $\boldsymbol{q}^a$ are given in the coordinates of frame $a$, also $\vec{k}$ must be expressed in the $a$ frame as the coordinate vector $\boldsymbol{k}^a$. It follows that

$$\boldsymbol{R}^a = \boldsymbol{I} + \sin\theta(\boldsymbol{k}^a)^\times + (1 - \cos\theta)(\boldsymbol{k}^a)^\times(\boldsymbol{k}^a)^\times \tag{77}$$

Next, the rotation is described in a frame $c$ as $\boldsymbol{p}^c = \boldsymbol{R}^c \boldsymbol{q}^c$. Then $\vec{k}$ must be described in the $c$ frame, which gives

$$\boldsymbol{R}^c = \boldsymbol{I} + \sin\theta(\boldsymbol{k}^c)^\times + (1 - \cos\theta)(\boldsymbol{k}^c)^\times(\boldsymbol{k}^c)^\times \tag{78}$$

The relation between the coordinate forms $\boldsymbol{R}^a$ and $\boldsymbol{R}^c$ is found from

$$\boldsymbol{p}^a = \boldsymbol{R}_c^a \boldsymbol{p}^c = \boldsymbol{R}_c^a \boldsymbol{R}^c \boldsymbol{q}^a = \boldsymbol{R}_c^a \boldsymbol{R}^c (\boldsymbol{R}_c^a)^{\mathrm{T}} \boldsymbol{q}^a \tag{79}$$

which shows that

$$\boldsymbol{R}^a = \boldsymbol{R}_c^a \boldsymbol{R}^c (\boldsymbol{R}_c^a)^{\mathrm{T}} \tag{80}$$

It is seen that rotation matrix transforms with a similarity transform, as opposed to a vector that transforms by a matrix multiplication, as in $\boldsymbol{k}^a = \boldsymbol{R}_c^a \boldsymbol{k}^c$.

Let the frame $b$ be given so that the rotation matrix $\boldsymbol{R}^a$ is the rotation from $a$ to $b$, that is, $\boldsymbol{R}^a = \boldsymbol{R}_b^a$. Then, if $\boldsymbol{p}^a = \boldsymbol{R}^a \boldsymbol{q}^a$, it follows that $\boldsymbol{q}^b = \boldsymbol{p}^a$. This means that the rotation from $a$ to $b$ is given by the rotation $\theta$ about $\boldsymbol{k}^a$, the rotation matrix is

$$\boldsymbol{R}_b^a = \boldsymbol{I} + \sin\theta(\boldsymbol{k}^a)^\times + (1 - \cos\theta)(\boldsymbol{k}^a)^\times(\boldsymbol{k}^a)^\times \tag{81}$$

It is straightforward to verify the $\boldsymbol{k}^b = \boldsymbol{k}^a$, which follows from $\boldsymbol{k}^b = \boldsymbol{R}_a^b \boldsymbol{k}^a = (\boldsymbol{R}_b^a)^{\mathrm{T}} \boldsymbol{k}^a$ and $(\boldsymbol{k}^a)^\times \boldsymbol{k}^a = \boldsymbol{0}$. This means that the expression (81) for $\boldsymbol{R}_b^a$ could have been written with $\boldsymbol{k}^b$ in place of $\boldsymbol{k}^a$.

Next, let the frame $d$ be defined so that the rotation matrix $\boldsymbol{R}^c$ is the rotation from $c$ to $d$, that is, $\boldsymbol{R}^c = \boldsymbol{R}_d^c$. Then, if $\boldsymbol{p}^c = \boldsymbol{R}^c \boldsymbol{q}^c$, it follows that $\boldsymbol{q}^d = \boldsymbol{p}^c$. This means the rotation matrix from $c$ to $d$ is

$$\boldsymbol{R}_d^c = \boldsymbol{I} + \sin\theta(\boldsymbol{k}^c)^\times + (1 - \cos\theta)(\boldsymbol{k}^c)^\times(\boldsymbol{k}^c)^\times \tag{82}$$

It is seen from (80) that

$$\boldsymbol{R}_b^a = \boldsymbol{R}_c^a \boldsymbol{R}_d^c (\boldsymbol{R}_c^a)^{\mathrm{T}} \tag{83}$$

Note that the two rotation matrices $\boldsymbol{R}_b^a$ and $\boldsymbol{R}_d^c$ describes the same physical rotation, which is a rotation $\theta$ about $\vec{k}$. The difference between the matrices is that they represent the rotation in two different coordinate frames.

**Example**

A rotation is described as a rotation by an angle $\theta = \pi/3$, which is $60^\circ$, about the vertical unit vector $\vec{k}$ through the origin. The frame $a$ is defined with a vertical $\boldsymbol{z}_a$ axis, so that $\vec{k}$ is along $\boldsymbol{z}_a$. The rotation described by $(\theta, \vec{k})$ rotates frame $a$ to a frame $b$. The rotation axis is given is the $c$ frame as $\boldsymbol{k}^a = [0, 0, 1]^{\mathrm{T}}$. The rotation matrix from $a$ to frame $b$ is then

$$\boldsymbol{R}_b^a = \boldsymbol{I} + \sin\theta(\boldsymbol{k}^a)^\times + (1-\cos\theta)(\boldsymbol{k}^a)^\times(\boldsymbol{k}^a)^\times = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{84}$$

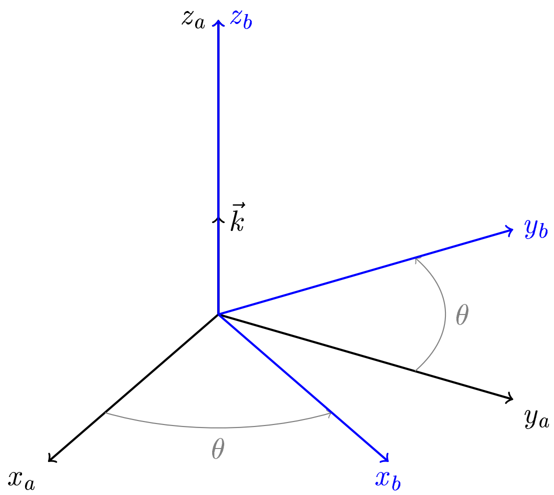



Figure 7: The rotation $(\theta, \vec{k})$ as a rotation from $a$ to $b$. Note that this is a simple rotation about the $z_a$ axis described by the rotation matrix $\boldsymbol{R}_b^a = \boldsymbol{R}_z(\theta)$.

A frame $c$ is defined with a vertical $x_c$ axis, with the $y_c$ axis along the $x_a$ axis, and the $z_c$ axis along the $y_a$ axis. The rotation matric from $a$ to $c$ is then

$$\boldsymbol{R}_c^a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{85}$$

where the columns are the coordinates of the unit vectors of the $c$ frame in the coordinates of the $a$ frame.

The rotation $(\theta, \vec{k})$ rotates the frame $c$ to the frame $d$. The rotation axis is given is the $c$ frame as $\boldsymbol{k}^c = [1, 0, 0]^{\mathrm{T}}$. The rotation matrix is then

$$\boldsymbol{R}_d^c = \boldsymbol{I} + \sin\theta(\boldsymbol{k}^c)^\times + (1-\cos\theta)(\boldsymbol{k}^c)^\times(\boldsymbol{k}^c)^\times = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{86}$$

Note that $\boldsymbol{R}_b^a$ and $\boldsymbol{R}_d^c$ describes the same rotation in two different frame combinations. The relation between the two rotation matrices is $\boldsymbol{R}_b^a = \boldsymbol{R}_c^a \boldsymbol{R}_d^c (\boldsymbol{R}_c^a)^{\mathrm{T}}$. This is verified by the calculation

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

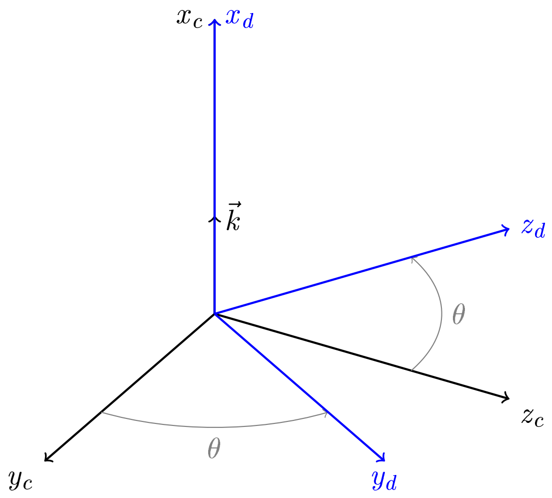



Figure 8: The rotation $(\theta, \vec{k})$ as a rotation from $c$ to $d$. This is a simple rotation about the $x_c$ axis described by the rotation matrix $\boldsymbol{R}_d^c = \boldsymbol{R}_x(\theta)$. Note that the rotation is given by the angle-axis parameters $(\theta, \vec{k})$, which are independent from the coordinate frame the rotation is described in. The rotation matrices $\boldsymbol{R}_b^a$ and $\boldsymbol{R}_d^c$ are therefore two alternative descriptions of the same rotation.

A physical interpretation of this would be to consider a rigid body $B$ that is rotated by an angle $\theta$ about the vector $\vec{k}$. Then, if a frame fixed in $B$ is aligned with $a$ before the rotation, then this frame will be rotated to frame $b$. In the same way a frame fixed in $B$ that is aligned with frame $c$ before the rotation will be rotated to frame $d$. The rotation of $B$ can then be described by the rotation from $a$ to $b$, or by the rotation from $c$ to $d$. □

### 3.8 Alternative derivation of the rotation matrix

In robotics textbooks like [16, 17] the rotation matrix $\boldsymbol{R}_k(\theta)$ of an angle $\theta$ about a unit vector $\boldsymbol{k}$ is derived by defining a coordinate frame with $z$ axis along the $\boldsymbol{k}$ vector, and then describing the rotation as a rotation about a $z$ axis. The rotation that is required to bring the $z$ axis in alignment with the rotation axis $\boldsymbol{k}$ is described as $\boldsymbol{R} = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_y(\beta)$ which is a rotation $\alpha$ about the $z$ axis followed by a rotation $\beta$ about the rotated $y$ axis. Then the rotation about $\boldsymbol{k}$ is given by

$$\boldsymbol{R}_k(\theta) = \boldsymbol{R}\boldsymbol{R}_z(\theta)\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_y(\beta)\boldsymbol{R}_z(\theta)\boldsymbol{R}_y(\beta)^{\mathrm{T}}\boldsymbol{R}_z(\alpha)^{\mathrm{T}} \tag{87}$$

since the rotation matrix transforms with a similarity transformation.

The angles are found by inspection of Figure 9 to be

$$\sin\alpha = \frac{k_y}{\sqrt{k_x^2 + k_y^2}}, \quad \cos\alpha = \frac{k_x}{\sqrt{k_x^2 + k_y^2}} \tag{88}$$

$$\sin\beta = \sqrt{k_x^2 + k_y^2}, \quad \cos\beta = k_z \tag{89}$$

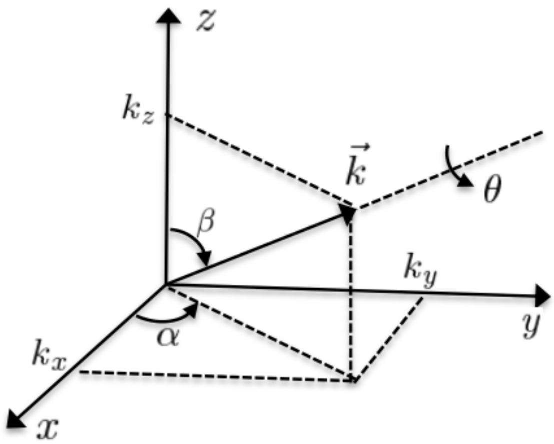



Figure 9: Rotation by an angle $\theta$ about $\vec{k}$.

Then $\boldsymbol{R} = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_y(\beta)$ is found to be

$$\boldsymbol{R} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \tag{90}$$

$$= \begin{bmatrix} \frac{k_x k_z}{\sqrt{k_x^2+k_y^2}} & -\frac{k_y}{\sqrt{k_x^2+k_y^2}} & k_x \\ \frac{k_y k_z}{\sqrt{k_x^2+k_y^2}} & \frac{k_x}{\sqrt{k_x^2+k_y^2}} & k_y \\ -\sqrt{k_x^2+k_y^2} & 0 & k_z \end{bmatrix} \tag{91}$$

Then, it is straightforward, but time consuming, to derive that

$$\boldsymbol{R}_k(\theta) = \begin{bmatrix} k_x^2(1-c_\theta)+c_\theta & k_x k_y(1-c_\theta)-k_z s_\theta & k_x k_z(1-c_\theta)+k_y s_\theta \\ k_x k_y(1-c_\theta)-k_z s_\theta & k_y^2(1-c_\theta)+c_\theta & k_y k_z(1-c_\theta)-k_x s_\theta \\ k_x k_z(1-c_\theta)-k_y s_\theta & k_y k_z(1-c_\theta)+k_x s_\theta & k_z^2(1-c_\theta)+c_\theta \end{bmatrix} \tag{92}$$

which can be written

$$\boldsymbol{R}_k(\theta) = \cos\theta \boldsymbol{I} + \sin(\theta)\boldsymbol{k}^\times + (1-\cos\theta)\boldsymbol{k}\boldsymbol{k}^\mathrm{T} \tag{93}$$

This the the rotation matrix in the form of (75). It is interesting to note that when the expression (68) is available, this derivation can be simplified. Instead of transforming the rotation matrix as in (87), the transformation can be done by transforming the $\boldsymbol{k}$ vector from being along the $z$ axis as $[0,0,1]^\mathrm{T}$ to $\boldsymbol{R}[0,0,1]^\mathrm{T} = [k_x, k_y, k_z]^\mathrm{T}$.

# 4 Dyadics

## 4.1 Introduction

A vector can be described by its coordinate-free form $\vec{v}$, which is given as a magnitude and a direction. The vector can alternatively be given in coordinate vector form as the column vector $\boldsymbol{v} = [v_1, v_2, v_3]^{\mathrm{T}}$. The coordinate vector form assumes that a coordinate frame has been defined so that $v_1, v_2, v_3$ are the coordinates of $\vec{v}$ in that frame.

In the case of the angle-axis description of a rotation the derivation was done using coordinate-free vectors, and it was found that the rotation of the vector $\vec{p}$ to the vector $\vec{q}$ was given by

$$\vec{q} = \vec{p} + \sin\theta \vec{k} \times \vec{p} + (1 - \cos\theta)\vec{k} \times (\vec{k} \times \vec{p})$$

Then, the expression for the rotation matrix was found by converting this equation to coordinate form $\boldsymbol{q}^a = \boldsymbol{R}^a \boldsymbol{p}^a$ where the coordinates of a frame $a$ is used. The coordinate vectors $\boldsymbol{p}^a$ and $\boldsymbol{q}^a$ can be transformed to another frame $c$ by multiplication with the rotation matrix $\boldsymbol{R}_a^c$, according to $\boldsymbol{p}^c = \boldsymbol{R}_a^c \boldsymbol{p}^a$ and $\boldsymbol{q}^c = \boldsymbol{R}_a^c \boldsymbol{q}^a$. It is interesting to note that also the rotation matrix can be transformed to frame $c$ by the similarity transformation $\boldsymbol{R}^c = \boldsymbol{R}_a^c \boldsymbol{R}^a \boldsymbol{R}_c^a$, as this gives $\boldsymbol{q}^c = \boldsymbol{R}^c \boldsymbol{p}^c$. This indicates that the rotation matrix satisfies certain coordinate-free properties that will be explored in the following.

It turns out that also the rotation matrix has a coordinate-free form. This coordinate-free form is called a dyadic, and in the following the necessary background for describing rotations with dyadics will be developed.

### 4.2 A basic dyadic

A dyadic is a pair of coordinate-free vectors, and the associated operation is the inner product. To be specific, let $\vec{a}$ and $\vec{b}$ be two vectors. Define the dyadic $\vec{D} = \vec{a}\vec{b}$, which is a pair of vectors. Then post-multiplication with a vector $\vec{c}$ is defined by

$$\vec{D} \cdot \vec{c} = (\vec{a}\vec{b}) \cdot \vec{c} = \vec{a}(\vec{b} \cdot \vec{c}) \tag{94}$$

where $\vec{b} \cdot \vec{c}$ is a scalar and $\vec{a}(\vec{b} \cdot \vec{c})$ is a vector. Pre-multiplication with a vector $\vec{c}$ is defined by

$$\vec{c} \cdot \vec{D} = \vec{c} \cdot (\vec{a}\vec{b}) = (\vec{c} \cdot \vec{a})\vec{b} \tag{95}$$

which is a vector.

A scalar result is obtained with post-multiplication with a vector $\vec{c}$ and pre-multiplication with a vector $\vec{d}$ according to

$$\vec{d} \cdot \vec{D} \cdot \vec{c} = \vec{d} \cdot (\vec{a}\vec{b}) \cdot \vec{c} = (\vec{d} \cdot \vec{a})(\vec{b} \cdot \vec{c}) \tag{96}$$

### 4.3 The matrix representation of a dyadic

Let $\vec{u}$ be a coordinate-free vector, and let $a$ be an arbitrary coordinate frame with orthogonal unit vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$. The vector $\vec{u}$ can be described in terms of its coordinates in $a$ as

$$\vec{u} = u_1\vec{a}_1 + u_2\vec{a}_2 + u_3\vec{a}_3$$

Let $\boldsymbol{u} = [u_1, u_2, u_3]^{\mathrm{T}}$ be a coordinate vector corresponding to $\vec{u}$, and let the matrix $\boldsymbol{D}$ be given by its elements $\{d_{ij}\}$. Suppose that the vector

$$\vec{w} = w_1\vec{a}_1 + w_2\vec{a}_2 + w_3\vec{a}_3$$

has a coordinate vector $\boldsymbol{w} = [w_1, w_2, w_3]^{\mathrm{T}}$ so that $\boldsymbol{w} = \boldsymbol{D}\boldsymbol{u}$. Note that this will be valid for any selection of coordinate frame $a$. This imposes certain conditions on the matrix $\boldsymbol{D}$.

The matrix multiplication is given by

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

and it is seen that the components of $\boldsymbol{w}$ are given in summation form as

$$w_i = \sum_{j=1}^{3} d_{ij} u_j \tag{97}$$

The coordinate free form is therefore

$$\vec{w} = \sum_{i=1}^{3} \sum_{j=1}^{3} d_{ij} u_j \vec{a}_i \tag{98}$$

Let the dyadic $\vec{D}$ be given by

$$\vec{D} = \sum_{i=1}^{3} \sum_{j=1}^{3} d_{ij} \vec{a}_i \vec{a}_j \tag{99}$$

Consider

$$\vec{w} = \vec{D} \cdot \vec{u} = \left( \sum_{i=1}^{3} \sum_{j=1}^{3} d_{ij} \vec{a}_i \vec{a}_j \right) \cdot \left( \sum_{k=1}^{3} u_k \vec{a}_k \right) = \sum_{i=1}^{3} \sum_{j=1}^{3} d_{ij} u_j \vec{a}_i \tag{100}$$

This shows that the following two operations are equivalent

$$\vec{w} = \vec{D} \cdot \vec{u} \quad \Leftrightarrow \quad \boldsymbol{w} = \boldsymbol{D}\boldsymbol{u} \tag{101}$$

Let $\boldsymbol{z} = [z_1, z_2, z_3]^{\mathrm{T}}$ be the column vector corresponding to $\vec{z}$. Next consider pre-multiplication

$$\vec{z} = \vec{u} \cdot \vec{D} = \left( \sum_{k=1}^{3} u_k \vec{a}_k \right) \cdot \left( \sum_{i=1}^{3} \sum_{j=1}^{3} d_{ij} \vec{a}_i \vec{a}_j \right) = \sum_{i=1}^{3} \sum_{j=1}^{3} d_{ij} u_i \vec{a}_j \tag{102}$$

Let $\boldsymbol{z} = [z_1, z_2, z_3]^{\mathrm{T}}$ be the column vector corresponding to $\vec{z}$. Then it follows that $\boldsymbol{z}^{\mathrm{T}} = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{D}$, which is the same as $\boldsymbol{z} = \boldsymbol{D}^{\mathrm{T}}\boldsymbol{u}$. This gives

$$\vec{z} = \vec{D} \cdot \vec{u} \quad \Leftrightarrow \quad \boldsymbol{z}^{\mathrm{T}} = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{D} \tag{103}$$

It is noted that the dyadic $\vec{A} = \vec{a}\vec{b}$ has the matrix form $\boldsymbol{A} = \boldsymbol{a}\boldsymbol{b}^{\mathrm{T}}$. This is verified by observing that $\vec{A} \cdot \vec{c} = \vec{a}(\vec{b} \cdot \vec{c})$ corresponds to $\boldsymbol{A}\boldsymbol{c} = \boldsymbol{a}(\boldsymbol{b}^{\mathrm{T}}\boldsymbol{c})$, and $\vec{c} \cdot \vec{A} = (\vec{c} \cdot \vec{a})\vec{b}$ corresponds to $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{A} = (\boldsymbol{c}^{\mathrm{T}}\boldsymbol{a})\boldsymbol{b}$. In particular, it is noted that $\vec{k}\vec{k}$ corresponds to $\boldsymbol{k}\boldsymbol{k}^{\mathrm{T}}$.

### 4.4 Coordinate transformations of a matrix representation of a dyadic

Let the frame $a$ have the orthogonal unit vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$, while the frame $b$ has the orthogonal unit vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3$. The dyadic can be expressed in the $a$ and $b$ frames as

$$\vec{D} = \sum_{i=1}^{3}\sum_{j=1}^{3} d_{ij}^a \vec{a}_i \vec{a}_j = \sum_{i=1}^{3}\sum_{j=1}^{3} d_{ij}^b \vec{b}_i \vec{b}_j \tag{104}$$

where $d_{ij}^a$ are the components in the $a$ frame and $d_{ij}^b$ are the components in the $b$ frame. The corresponding matrix representations are

$$\boldsymbol{D}^a = \{d_{ij}^a\}, \quad \boldsymbol{D}^b = \{d_{ij}^b\} \tag{105}$$

Consider the equation $\vec{w} = \vec{D} \cdot \vec{u}$, which is in dyadic form. The corresponding matrix form can be given in the $a$ and $b$ frames as

$$\boldsymbol{w}^a = \boldsymbol{D}^a \boldsymbol{u}^a, \quad \boldsymbol{w}^b = \boldsymbol{D}^b \boldsymbol{u}^b \tag{106}$$

where $\boldsymbol{w}^a = \boldsymbol{R}_b^a \boldsymbol{w}^b$ and $\boldsymbol{u}^a = \boldsymbol{R}_b^a \boldsymbol{u}^b$. It follows that

$$\boldsymbol{w}^a = \boldsymbol{R}_b^a \boldsymbol{D}^b \boldsymbol{u}^b = \boldsymbol{R}_b^a \boldsymbol{D}^b \boldsymbol{R}_a^b \boldsymbol{u}^a \tag{107}$$

and it is concluded that the matrix representation of the dyadic $\vec{D}$ transforms according to

$$\boldsymbol{D}^a = \boldsymbol{R}_b^a \boldsymbol{D}^b \boldsymbol{R}_a^b \tag{108}$$

Note that the coordinate vectors transform by matrix multiplication, while the matrix representation transforms with a similarity transform.

### 4.5 The skew-symmetric dyadic

The skew-symmetric dyadic of a vector $\vec{v} = v_1\vec{a}_1 + v_2\vec{a}_2 + v_3\vec{a}_3$ is defined by

$$\vec{v}^{\times} = -v_3\vec{a}_1\vec{a}_2 + v_2\vec{a}_1\vec{a}_3 + v_3\vec{a}_2\vec{a}_1 - v_1\vec{a}_2\vec{a}_3 - v_2\vec{a}_3\vec{a}_1 + v_1\vec{a}_3\vec{a}_2 \tag{109}$$

It follows that

$$\vec{w} = \vec{v} \times \vec{u} = \vec{v}^{\times} \cdot \vec{u} \tag{110}$$

for an arbitrary vector $\vec{u}$. It is seen that the matrix form of $\vec{v}^{\times}$ is the skew-symmetric matrix $\boldsymbol{v}^{\times}$, which gives the corresponding matrix form

$$\boldsymbol{w} = \boldsymbol{v}^{\times} \boldsymbol{u} \tag{111}$$

In the two frames $a$ and $b$ this is written $\boldsymbol{w}^a = (\boldsymbol{v}^a)^{\times} \boldsymbol{u}^a$ and $\boldsymbol{w}^b = (\boldsymbol{v}^b)^{\times} \boldsymbol{u}^b$. The coordinate transformation for the skew-symmetric matrix is therefore

$$(\boldsymbol{v}^a)^{\times} = \boldsymbol{R}_b^a (\boldsymbol{v}^b)^{\times} \boldsymbol{R}_a^b \tag{112}$$

Moreover, note that $(\boldsymbol{R}_b^a \boldsymbol{v}^b)^{\times} = \boldsymbol{R}_b^a (\boldsymbol{v}^b)^{\times} \boldsymbol{R}_a^b$.

### 4.6 The triple vector product

The triple vector product can be written in dyadic form as

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{a}^{\times} \cdot \vec{b}^{\times} \cdot \vec{c}$$

This can be combined with

$$\vec{b}\vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b}\vec{c} = \left(\vec{b}\vec{a} - (\vec{a} \cdot \vec{b})\vec{I}\right) \vec{c}$$

to get the result

$$\vec{a}^{\times} \cdot \vec{b}^{\times} = \vec{b}\vec{a} - (\vec{a} \cdot \vec{b})\vec{I} \tag{113}$$

In particular, it is noted that for a unit vector $\vec{k}$,

$$\vec{k}^{\times} \cdot \vec{k}^{\times} = \vec{k}\vec{k} - \vec{I} \tag{114}$$

**Example**

Consider the calculation

$$\left(\vec{a} \times \vec{b}\right) \cdot \left(\vec{c} \times \vec{d}\right) = \left(\vec{a} \cdot \vec{b}^{\times}\right) \cdot \left(\vec{c}^{\times} \cdot \vec{d}\right) = \vec{a} \cdot \left(\vec{b}^{\times} \cdot \vec{c}^{\times}\right) \cdot \vec{d} \tag{115}$$

This can be further developed as

$$\left(\vec{a} \times \vec{b}\right) \cdot \left(\vec{c} \times \vec{d}\right) = \vec{a} \cdot \left(\vec{c}\vec{b} - \vec{b} \cdot \vec{c}\right) \cdot \vec{d} \tag{116}$$

### 4.7 Quadratic forms

The kinetic energy due to the rotation of a rigid body is the quadratic form

$$K = \frac{1}{2}\vec{\omega} \cdot \vec{M} \cdot \vec{\omega} = \frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{\omega} \tag{117}$$

The inertia dyadic and the inertial matrix are related by

$$\vec{M} = \sum_{i=1}^{3}\sum_{j=1}^{3} m_{ij}\vec{a}_i\vec{a}_j, \quad \boldsymbol{M} = \{m_{ij}\} \tag{118}$$

where $a$ is an arbitrary frame with orthogonal unit vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$. The angular velocity is given by $\vec{\omega} = \omega_1\vec{a}_1 + \omega_2\vec{a}_2 + \omega_3\vec{a}_3$ with coordinate form $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]^{\mathrm{T}}$. The coordinate frame $a$ is often selected as a frame that is fixed in the rigid body. In this case the inertia dyadic $\vec{M}$ and the inertia matrix $\boldsymbol{M}$ are constant.

In the calculation of the inertial dyadic, the expression

$$K = \int_B (\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r}) dm \tag{119}$$

occurs. This can be developed using

$$(\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r}) = -(\vec{\omega} \times \vec{r}) \cdot (\vec{r} \times \vec{\omega}) = -\vec{\omega} \cdot \left(\vec{r}^{\times} \cdot \vec{r}^{\times}\right) \cdot \vec{\omega} \tag{120}$$

or alternatively as

$$(\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r}) = \left(\vec{r} \cdot \vec{r}\vec{I} - \vec{r}\vec{r}\right) \cdot \vec{\omega} \tag{121}$$

Then the kinetic energy can be found from the well-known formula

$$K = \frac{1}{2}\vec{\omega} \cdot \vec{M} \cdot \vec{\omega} = \frac{1}{2}\vec{\omega} \cdot \int_B \left(\vec{r} \cdot \vec{r}\vec{I} - \vec{r}\vec{r}\right) dm \cdot \vec{\omega} \tag{122}$$

### 4.8 The identity dyadic

The identity dyadic is defined by

$$\vec{I} = \sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij}\vec{a}_i\vec{a}_j = \vec{a}_1\vec{a}_1 + \vec{a}_2\vec{a}_2 + \vec{a}_3\vec{a}_3 \tag{123}$$

It is straightforward top show that

$$\vec{I}\cdot\vec{u} = \vec{u}\cdot\vec{I} = \vec{u} \tag{124}$$

The matrix expression is $\boldsymbol{I}\boldsymbol{u} = \boldsymbol{u}$. Moreover,

$$\vec{I}\cdot\vec{D} = \vec{D}\cdot\vec{I} = \vec{D} \tag{125}$$

for any dyadic $\vec{D}$. The corresponding matrix formulation is $\boldsymbol{I}\boldsymbol{D} = \boldsymbol{D}\boldsymbol{I} = \boldsymbol{D}$.

### 4.9 The rotation dyadic

Consider the rotation from an arbitrary frame $a$ with orthogonal unit vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ to a frame $b$ with orthogonal unit vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3$. Consider the dyadic given by

$$\vec{R} = \vec{b}_1\vec{a}_1 + \vec{b}_2\vec{a}_2 + \vec{b}_3\vec{a}_3 \tag{126}$$

Insertion of

$$\vec{a}_1 = \vec{b}_1(\vec{a}_1\cdot\vec{b}_1) + \vec{b}_2(\vec{a}_1\cdot\vec{b}_2) + \vec{b}_3(\vec{a}_1\cdot\vec{b}_3)$$

and the corresponding expressions for $\vec{a}_2$ and $\vec{a}_3$ gives

$$\vec{R} = \vec{b}_1\vec{b}_1(\vec{a}_1\cdot\vec{b}_1) + \vec{b}_1\vec{b}_2(\vec{a}_1\cdot\vec{b}_2) + \vec{b}_1\vec{b}_3(\vec{a}_1\cdot\vec{b}_3) + \ldots \tag{127}$$

which gives

$$\vec{R} = \sum_{i=1}^{3}\sum_{j=1}^{3}(\vec{a}_i\cdot\vec{b}_j)\vec{b}_i\vec{b}_j \tag{128}$$

The matrix representation of the dyadic $\vec{R}$ in the coordinates of $b$ is therefore

$$\boldsymbol{R}^b = \{\vec{a}_i\cdot\vec{b}_j\} = \boldsymbol{R}_b^a \tag{129}$$

which is the rotation matrix from $a$ to $b$. It follows that $\vec{R}$ is the rotation dyadic from $a$ to $b$.

This is supported by calculating

$$\vec{R}\cdot\vec{p} = (\vec{b}_1\vec{a}_1 + \vec{b}_2\vec{a}_2 + \vec{b}_3\vec{a}_3)\cdot(p_1\vec{a}_1 + p_2\vec{a}_2 + p_3\vec{a}_3) = p_1\vec{b}_1 + p_2\vec{b}_2 + p_3\vec{b}_3 \tag{130}$$

which shows that if

$$\vec{p} = p_1\vec{a}_1 + p_2\vec{a}_2 + p_3\vec{a}_3 \tag{131}$$

then $\vec{q} = \vec{R}\cdot\vec{p}$ will be

$$\vec{q} = p_1\vec{b}_1 + p_2\vec{b}_2 + p_3\vec{b}_3 \tag{132}$$

This means that the coordinates of $\vec{q} = \vec{R}\cdot\vec{p}$ in $b$ are equal to the coordinates of $\vec{p}$ in $a$. This is consistent with the active form of the rotation matrix, where a vector $\boldsymbol{q}^a$ given by $\boldsymbol{q}^a = \boldsymbol{R}_b^a\boldsymbol{p}^a$ will satisfy $\boldsymbol{q}^b = \boldsymbol{p}^a$.

### 4.10 The rotation dyadic in terms of the angle and axis parameters

The simple rotation of a vector $\vec{p}$ to a vector $\vec{q}$ by an angle $\theta$ about a fixed unit vector $\vec{k}$ can be written

$$\vec{q} = (\vec{I} + \sin\theta \vec{k}^{\times} + (1 - \cos\theta)\vec{k}^{\times} \cdot \vec{k}^{\times}) \cdot \vec{p} = \vec{R} \cdot \vec{p} \tag{133}$$

where

$$\vec{R} = \vec{I} + \sin\theta \vec{k}^{\times} + (1 - \cos\theta)\vec{k}^{\times} \cdot \vec{k}^{\times} \tag{134}$$

is the rotation dyadic.

The coordinate form in an arbitrary frame $a$ is $\boldsymbol{q}^a = \boldsymbol{R}^a \boldsymbol{p}^a$, where the $\boldsymbol{R}^a$ is the matrix form $\vec{R}$ in frame $a$, which is

$$\boldsymbol{R}^a = \boldsymbol{I} + \sin\theta (\boldsymbol{k}^a)^{\times} + (1 - \cos\theta)(\boldsymbol{k}^a)^{\times}(\boldsymbol{k}^a)^{\times} \tag{135}$$

This means that the rotation matrix is a matrix representation of the rotation dyadic.

It follows that the rotation matrix transforms between different coordinate systems with a similarity transform. The rotation matrix $\boldsymbol{R}^a$ is therefore transformed to a frame $c$ with the similarity transform

$$\boldsymbol{R}^c = \boldsymbol{R}^c_a \boldsymbol{R}^a \boldsymbol{R}^a_c \tag{136}$$

This agrees with

$$\boldsymbol{R}^c = \boldsymbol{I} + \sin\theta (\boldsymbol{k}^c)^{\times} + (1 - \cos\theta)(\boldsymbol{k}^c)^{\times}(\boldsymbol{k}^c)^{\times} \tag{137}$$

and the transformation rule

$$(\boldsymbol{k}^c)^{\times} = \boldsymbol{R}^c_a (\boldsymbol{k}^a)^{\times} \boldsymbol{R}^a_c \tag{138}$$

for the skew-symmetric matrix.

It follows that the rotation of the vector $\vec{p}$ to $\vec{q}$ can be described in terms of either of the following equivalent expressions

$$\vec{q} = \vec{R} \cdot \vec{p} \quad \Leftrightarrow \quad \boldsymbol{q}^a = \boldsymbol{R}^a \boldsymbol{p}^a \quad \Leftrightarrow \quad \boldsymbol{q}^c = \boldsymbol{R}^c \boldsymbol{p}^c \tag{139}$$

## 5 Euler angles

### 5.1 ZYZ Euler angles

In many applications the rotation matrix is described in terms of 3 Euler angles. This is a sequence of 3 simple rotations about the coordinate axes. Different Euler angle combinations can be obtained depending on the ordering of the rotations.

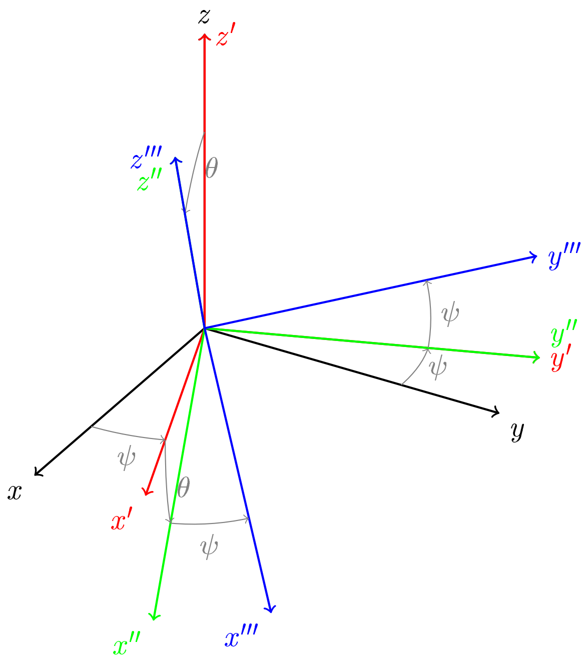
A common Euler angle convention is to represent a rotation matrix as

$$\boldsymbol{R}_{\mathrm{ZYZ}} = \boldsymbol{R}_z(\psi)\boldsymbol{R}_y(\theta)\boldsymbol{R}_z(\phi) \tag{140}$$

Then the rotation $\boldsymbol{R}^1_2 = \boldsymbol{R}_{\mathrm{ZYZ}}$ from frame 1 to 2 is described as a sequence of rotations where the first rotation is from frame 1 to frame $1'$, and is described by an angle $\phi$ about the $z_1$ axis. The next rotation is from frame $1'$ to $1''$ about the current $y$ axis, which is the $y_{1'}$ axis. The last rotation is from frame 1″ to frame 2 described by an angle $\psi$ about the current $z$ axis, which is the $z_{1''}$ axis. The resulting rotation matrix is

$$\boldsymbol{R}_{\mathrm{ZYZ}} = \begin{bmatrix} c_\psi c_\theta c_\phi - s_\psi s_\phi & -c_\psi c_\theta s_\phi - s_\psi c_\phi & c_\psi s_\theta \\ s_\psi c_\theta c_\phi + c_\psi s_\phi & -s_\psi c_\theta s_\phi - c_\psi c_\phi & s_\psi s_\theta \\ -s_\theta c_\phi & s_\theta s_\phi & c_\theta \end{bmatrix} \quad (141)$$

Note that the rotations are done about the current rotated axes, and that the sequence of the rotation matrices in the expression (140) is the same as the sequence of the rotations.




Let the elements be denoted $\boldsymbol{R}_{\mathrm{ZYZ}} = \{r_{ij}\}$. Then, if it is assumed that $s_\theta > 0$, the first angle can be computed as $\psi = \mathrm{Atan2}(r_{23}, r_{13})$. It is necessary that $r_{23}$ and $r_{13}$ are not both equal to zero, as $\mathrm{Atan2}(0,0)$ is undefined. This will only occur when $s_\theta = 0$, which means that there is no unique solution as the rotations $\psi$ and $\phi$ are about the same axis. The corresponding angle $\theta$ is then computed as $\theta = \mathrm{Atan2}(\sqrt{r_{13}^2 + r_{23}^2}, r_{33})$, and $\phi = \mathrm{Atan2}(r_{23}, -r_{31})$.

To include the possibility that $s_\theta < 0$ a logical value $F$ can be used, where $F = 1$ when $s_\theta > 0$ and $F = -1$ when $s_\theta < 0$. Then a consistent solution is found with

$$\psi = \mathrm{Atan2}(F r_{23}, F r_{13}) \quad (142)$$

$$\theta = \mathrm{Atan2}(F\sqrt{r_{13}^2 + r_{23}^2}, r_{33}) \quad (143)$$

$$\phi = \mathrm{Atan2}(F r_{32}, -F r_{31}) \quad (144)$$

The second angle can alternatively be computed from $\theta = \mathrm{Atan2}(F\sqrt{1 - r_{33}^2}, r_{33})$.

**Example**

```
% Calculation of rotation matrix from ZYZ Euler angles,
% and calculation of Euler angles from the rotation matrix.
```

```
% Input Euler angles
psi = 0.3; theta = 0.2; phi = 0.1;
% Calculate rotation matrix
R1 = AngleAxis2R(psi, [0;0;1]);
R2 = AngleAxis2R(theta, [0;1;0]);
R3 = AngleAxis2R(phi, [0;0;1]);
R = R1*R2*R3

% Calculate Euler angles from rotation matrix
F = 1; % Select solution with F = 1 or F = -1
a = atan2(F*R(2,3),F*R(1,3))
b = atan2(F*sqrt(R(1,3)^2+R(2,3)^2),R(3,3))
c = atan2(F*R(3,2),-F*R(3,1))

% Test is solution is correct
R1 = AngleAxis2R(a, [0;0;1]);
R2 = AngleAxis2R(b, [0;1;0]);
R3 = AngleAxis2R(c, [0;0;1]);
Rt = R1*R2*R3;
DeltaR = Rt-R % Zero matrix expected

%%%%%%%%%%%

function R = AngleAxis2R(theta, k)
k = k/norm(k);
kskew = [0 -k(3) k(2) ; k(3) 0 -k(1); -k(2) k(1) 0];
R = cos(theta) * eye(3) + sin(theta)*kskew + (1-cos(theta))*k*k';
```

### 5.2 Roll-pitch-yaw Euler angles

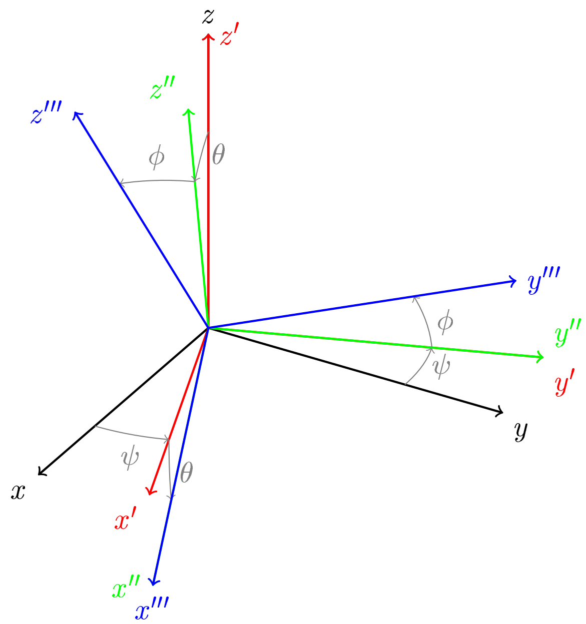
An alternative Euler angle convention is given by the roll-pitch-yaw angles as

$$\boldsymbol{R}_{\mathrm{ZYX}} = \boldsymbol{R}_z(\psi)\boldsymbol{R}_y(\theta)\boldsymbol{R}_x(\phi) \tag{145}$$

In this case, the rotation $\boldsymbol{R}_2^1 = \boldsymbol{R}_{\mathrm{ZYX}}$ is a yaw angle $\psi$ about the $z_1$ axis, followed by a pitch angle $\theta$ about the $y_{1'}$ axis, and finally a roll angle $\phi$ about the $x_{1''}$ axis. The resulting rotation matrix is

$$\boldsymbol{R}_{\mathrm{ZYX}} = \begin{bmatrix} c_\psi c_\theta & c_\psi s_\theta s_\phi - s_\psi c_\phi & c_\psi s_\theta c_\phi + s_\psi s_\phi \\ s_\psi c_\theta & s_\psi s_\theta s_\phi + c_\psi c_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix} \tag{146}$$

Let the elements of the rotation matrix be given as $\boldsymbol{R}_{\mathrm{ZYX}} = \{r_{ij}\}$. Define the variable $E$ so that $E = 1$ when $c_\theta > 0$, and $E = -1$ when $c_\theta < 0$. Then the first angle can be computed as $\psi = \mathrm{Atan2}(Er_{21}, Er_{11})$. It is necessary that $r_{21}$ and $r_{11}$ are not both equal to zero, as $\mathrm{Atan2}(0,0)$ is undefined. This will only occur when $c_\theta = 0$, which means that there is no unique solution as the rotations $\psi$ and $\phi$ are about the same axis. The corresponding angle $\theta$ is then computed as $\theta = \mathrm{Atan2}(-r_{31}, E\sqrt{r_{32}^2 + r_{33}^2})$, and $\phi = \mathrm{Atan2}(Er_{32}, Er_{33})$.




To sum up, the two possible solutions are

$$\psi = \text{Atan2}(E r_{21}, E r_{11}) \tag{147}$$

$$\theta = \text{Atan2}(-r_{31}, E\sqrt{r_{32}^2 + r_{33}^2}) \tag{148}$$

$$\phi = \text{Atan2}(E r_{32}, E r_{33}) \tag{149}$$

where $E$ is selected as $E = 1$ or $E = -1$ to give the desired solution. It is note that the second angle can alternatively be computed from $\theta = \text{Atan2}(-r_{31}, E\sqrt{1 - r_{31}^2})$, which is consistent with $r_{31}^2 + r_{32}^2 + r_{33}^2 = 1$, which follows from the orthogonality of the rotation matrix.

### 5.3 Euler angles about fixed axes

In this section a rotation $\boldsymbol{R}_2^1$ from frame 1 to 2 is described by Euler angles, where the rotations are about fixed axes, as opposed to rotations about current axes, as presented above. A sequence XYZ will be considered where the first rotation is a rotation $\phi$ about the $x_1$ axis, the a rotation $\theta$ about the $y_1$ axis, and finally a rotation $\psi$ about the $z_1$ axis. After the first rotation, the frame 1 has been rotated to frame $1'$ by $\boldsymbol{R}_x(\phi)$. The next rotation $\boldsymbol{R}_y(\theta)$ about the $y_1$ axis must then be transformed to the coordinates of the $1'$ frame. A vector would transform by $\boldsymbol{v}^{1'} = \boldsymbol{R}_1^{1'}\boldsymbol{v}^1 = \boldsymbol{R}_x(\phi)^{\mathrm{T}}\boldsymbol{v}^1$ while a rotation matrix is transformed from 1 to $1'$ by $\boldsymbol{R}_x(\phi)^{\mathrm{T}}\boldsymbol{R}_y(\theta)\boldsymbol{R}_x(\phi)$. The last rotation must be transformed to the coordinates of frame $1''$, which is done with $\boldsymbol{R}_x(\phi)^{\mathrm{T}}\boldsymbol{R}_y(\theta)^{\mathrm{T}}\boldsymbol{R}_z(\psi)\boldsymbol{R}_y(\theta)\boldsymbol{R}_x(\phi)$. The resulting rotation matrix is

$$\boldsymbol{R}_{\text{XYZ,fixed}} = \boldsymbol{R}_x(\phi)[\boldsymbol{R}_x(\phi)^{\mathrm{T}}\boldsymbol{R}_y(\theta)\boldsymbol{R}_x(\phi)][\boldsymbol{R}_x(\phi)^{\mathrm{T}}\boldsymbol{R}_y(\theta)^{\mathrm{T}}\boldsymbol{R}_z(\psi)\boldsymbol{R}_y(\theta)\boldsymbol{R}_x(\phi)] \tag{150}$$

Straightforward cancellations of the type $\boldsymbol{R}_x(\phi)\boldsymbol{R}_x(\phi)^{\mathrm{T}} = \boldsymbol{I}$ and $\boldsymbol{R}_y(\theta)\boldsymbol{R}_y(\theta)^{\mathrm{T}} = \boldsymbol{I}$ lead to the result

$$\boldsymbol{R}_{\text{XYZ,fixed}} = \boldsymbol{R}_z(\psi)\boldsymbol{R}_y(\theta)\boldsymbol{R}_x(\phi) \tag{151}$$

It is seen that $\boldsymbol{R}_{\mathrm{XYZ,fixed}} = \boldsymbol{R}_{\mathrm{ZYX}}$. This means that the rotation matrix for a sequence of Euler angles about fixed axes is the same as the rotation matrix for same Euler angles about current axes where the rotations are done in the reverse sequence.

### 5.4 ISO Euler angles

The ISO 9787 standard for robots and robotic devices describes a rotation $\boldsymbol{R}_{\mathrm{ISO}} = \boldsymbol{R}_2^1$ from frame 1 to 2 by Euler angles about axes of rotation that are fixed in frame 1. The standard specifies that there a rotation $A$ about the $x_1$ axis, a rotation $B$ about the $y_1$ axis, and a rotation $C$ about the $z_1$ axis. The explicit sequence of rotations is not specified in the standard, and must be selected as the Euler angles do not commute. It is reasonable to select the sequence in alphabetical order as first a rotation $A$ about $x_1$, then $B$ about the fixed $y_1$ axis, and finally $C$ about the fixed $z_1$ axis. The resulting rotation matrix is

$$\boldsymbol{R}_{\mathrm{ISO}} = \boldsymbol{R}_z(C)\boldsymbol{R}_y(B)\boldsymbol{R}_x(A) \tag{152}$$

where the sequence of the rotation matrices is the opposite of the sequence of rotations, since the rotations are about fixed axes.

### 5.5 KUKA Euler angles

The robot controllers of KUKA use rotations about current axes. The sequence is first a rotation $A$ about the $z$ axis, then a rotation $B$ about the current $y$ axis, and finally a rotation $C$ about the current rotated $x$ axis. The resulting rotation matrix is

$$\boldsymbol{R}_{\mathrm{KUKA}} = \boldsymbol{R}_z(A)\boldsymbol{R}_y(B)\boldsymbol{R}_x(C) \tag{153}$$

## 6 Forward kinematics

### 6.1 Introduction

A typical industrial robot will have a manipulator where there are $n$ links that are connected with joints in a serial arrangement so that the link $i-1$ is connected with link $i$ with joint $i$. The fixed base is denoted as link 0, and is connected with joint 1 to link 1. The final link is the hand or end effector, which is link $n$. Each joint has one degree of freedom, which can be a rotation or a translation. This means that a joint will either be a rotational joint with a rotation about the joint axis, or a it can be a prismatic joint with a translation along the joint axis. The manipulator will then have $n$ joint variables, with one joint variable for each joint. The joint variable will be an angle for a rotational joint, and a translation for a prismatic joint. The position and orientation of the robot hand, which is also called the end effector, will depend on the joint variables.

The position and orientation of the end effector relative to the base can be described with a homogeneous transformation matrix, which will be presented in the following. The computation of the position and orientation of the end effector given the joint variables is called the forward kinematics problem. In the following the forward kinematic problem will be discussed, and the necessary tools and methods will be introduced and explained.

### 6.2 Homogeneous transformation matrices

In previous sections the rotation matrix has been introduced to describe the orientation of a coordinate frame with respect to some other frame. In addition it is necessary to describe the translational position of a coordinate frame relative to another frame. To do this the concept of a *homogeneous transformation matrix* is introduced. This is a matrix that describes the displacement of a coordinate frame with respect to a reference frame in terms of the difference in position and orientation between the two frames.

Consider a frame $a$ and a frame $b$, and let $\boldsymbol{R}_b^a$ be the rotation matrix from $a$ to $b$, while $\boldsymbol{r}_{ab}^a$ is the position in $a$ coordinates of the origin of frame $b$ relative to the origin of frame $a$. Then the position and orientation of frame $b$ relative to frame $a$ is given by the homogeneous transformation matrix

$$\boldsymbol{T}_b^a = \begin{bmatrix} \boldsymbol{R}_b^a & \boldsymbol{r}_{ab}^a \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix} \in SE(3) \tag{154}$$

which is also called the displacement from $a$ to $b$.

Here the set $SE(3)$ is the Special Euclidean Group of dimension 3 defined by

$$SE(3) = \left\{ \boldsymbol{T} \mid \boldsymbol{T} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{r} \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix},\ \boldsymbol{R} \in SO(3),\ \boldsymbol{r} \in \mathbb{R}^3 \right\} \tag{155}$$

The inverse of $\boldsymbol{T}_b^a$ is found by matrix inversion to be

$$(\boldsymbol{T}_b^a)^{-1} = \begin{bmatrix} (\boldsymbol{R}_b^a)^{\mathrm{T}} & -(\boldsymbol{R}_b^a)^{\mathrm{T}} \boldsymbol{r}_{ab}^a \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_a^b & \boldsymbol{r}_{ba}^b \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix} = \boldsymbol{T}_a^b \tag{156}$$

This means that

$$(\boldsymbol{T}_b^a)^{-1} = \boldsymbol{T}_a^b \tag{157}$$

It is seen that the inverse of the displacement from $a$ to $b$ in terms of the homogeneous transformation matrix $\boldsymbol{T}_b^a$ is $\boldsymbol{T}_a^b$, which is the displacement from $b$ to $a$.

The displacement from $a$ to $c$ can be described as a composite displacement

$$\boldsymbol{T}_c^a = \boldsymbol{T}_b^a \boldsymbol{T}_c^b \tag{158}$$

where a displacement from $a$ to $b$ is followed by a displacement from $b$ to $c$. This is seen from

$$\begin{aligned} \boldsymbol{T}_b^a \boldsymbol{T}_c^b &= \begin{bmatrix} \boldsymbol{R}_b^a & \boldsymbol{r}_{ab}^a \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{R}_c^b & \boldsymbol{r}_{bc}^b \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{R}_b^a \boldsymbol{R}_c^b & \boldsymbol{r}_{ab}^a + \boldsymbol{R}_b^a \boldsymbol{r}_{bc}^b \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{R}_c^a & \boldsymbol{r}_{ac}^a \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix} \\ &= \boldsymbol{T}_c^a \end{aligned} \tag{159}$$

It is straightforward to extend this result to

$$\boldsymbol{T}_d^a = \boldsymbol{T}_b^a \boldsymbol{T}_c^b \boldsymbol{T}_d^c \tag{160}$$

and so on.

### 6.3 The homogeneous transformation matrix of a simple rotation

The homogeneous transformation matrix describing a simple rotation by an angle $\theta$ about a fixed axis $\boldsymbol{k}$ is given by

$$\mathbf{Rot}_k(\theta) = \begin{bmatrix} \boldsymbol{R}_k(\theta) & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix} \tag{161}$$

where $\boldsymbol{R}_k(\theta) = \boldsymbol{I} + \sin\theta \boldsymbol{k}^{\times} + (1 - \cos\theta)\boldsymbol{k}^{\times}\boldsymbol{k}^{\times}$ is the rotation matrix of the simple rotation. Simple rotations about the $x$ and $z$ axes are described by the homogeneous transformation matrices

$$\mathbf{Rot}_x(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\phi & -s_\phi & 0 \\ 0 & s_\phi & c_\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{Rot}_z(\psi) = \begin{bmatrix} c_\psi & -s_\psi & 0 & 0 \\ s_\psi & c_\psi & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\boldsymbol{x} = [1, 0, 0]^{\mathrm{T}}$ and $\boldsymbol{z} = [0, 0, 1]^{\mathrm{T}}$.

### 6.4 The homogeneous transformation matrix of a translation

A displacement which is a translation about a fixed axis will be a displacement with no rotation. The resulting homogeneous transformation matrix is written

$$\mathbf{Trans}_k(h) = \begin{bmatrix} \boldsymbol{I} & h\boldsymbol{k} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix} \tag{162}$$

Translations along the $x$ and $z$ axes will be described by the homogeneous transformation matrices

$$\mathbf{Trans}_x(a) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{Trans}_z(d) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is straightforward to verify that

$$\mathbf{Trans}_k(h)\mathbf{Rot}_k(\theta) = \mathbf{Rot}_k(\theta)\mathbf{Trans}_k(h) \tag{163}$$

which follows from $\boldsymbol{R}_k(\theta)\boldsymbol{k} = \boldsymbol{k}$. This rotation about a fixed axis and translation along the same axis will commute.

### 6.5 The displacement of the end effector

The displacement of the end effector relative to the base is described with a homogeneous transformation matrix which is a composite displacement of the $n$ displacements of the manipulator joints. The fixed base frame is denoted as frame 0. Then frame 1 is fixed in link 1, frame 2 is fixed in link 2 and so on up to frame $n$ which is fixed in the final link, which is link $n$. The displacement of frame $i$ relative to frame $i-1$ is denoted $\boldsymbol{T}_i^{i-1}$ for $i = 1, \ldots, n$, where $\boldsymbol{T}_i^{i-1}$ is called the link displacement. The end effector displacement is given by the composite displacement

$$\boldsymbol{T}_n^0 = \boldsymbol{T}_1^0 \boldsymbol{T}_2^1 \ldots \boldsymbol{T}_n^{n-1} \tag{164}$$

This means that is the link displacements are available, the end effector displacement can be computed by simply multiplying the link displacements. In thye next section it will be shown how this can be done.

### 6.6 The Denavit-Hartenberg convention

The Denavit-Hartenberg convection is based on the article [4], and is standard material in robotics textbooks like [17, 16]. The convention is based on describing a homogeneous transformation as a composite displacement of 4 homogeneous transformation matrices, starting with a rotation $\theta$ about the $z$ axis and then a translation $d$ along the same axis. This is followed by a rotation $\alpha$ about the $x$ axis followed by a translation $a$ along the same axis. This gives a homogeneous transformation matrix that is described in terms of the 4 parameters $\theta$, $d$, $\alpha$ and $a$, which are called the Denavit-Hartenberg parameters of the displacement. The resulting displacement is given by

$$\boldsymbol{T}_i^{i-1} = \mathbf{Rot}_z(\theta_i)\mathbf{Trans}_z(d_i)\mathbf{Rot}_x(\alpha_i)\mathbf{Trans}_x(a_i) \tag{165}$$

The rotation and translation about the $z$ axis gives the composite displacement

$$\mathbf{Rot}_z(\theta_i)\mathbf{Trans}_z(d_i) = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{166}$$

The rotation and translation about the $x$ axis gives the composite displacement

$$\mathbf{Rot}_x(\alpha_i)\mathbf{Trans}_x(a_i) = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{167}$$

This gives the result

$$\boldsymbol{T}_i^{i-1} = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{168}$$

According to the convention, the $z_i$ axis is assigned to the joint axis. This means that if joint $i$ is a rotational joint, then the joint variable is the angle $\theta_i$, which is the rotation angle about the $z_i$ axis. If joint $i$ is a prismatic joint, then the joint variable is $d_i$, which is the translation along the $z_i$ axis.

### 6.7 Computation of the forward kinematics

The forward kinematics of a manipulator is given by its Denavit-Hartenberg parameters. These parameters are typically presented in the form

| Link | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_{i0}$ |
|---|---|---|---|---|
| 1 | $a_1$ | $\alpha_1$ | $d_1$ | $\theta_1$ |
| 2 | $a_2$ | $\alpha_2$ | $d_2$ | $\theta_2$ |
| 3 | $a_3$ | $\alpha_3$ | $d_3$ | $\theta_3$ |
| 4 | $a_4$ | $\alpha_4$ | $d_4$ | $\theta_4$ |
| 5 | $a_5$ | $\alpha_5$ | $d_5$ | $\theta_5$ |
| 6 | $a_6$ | $\alpha_6$ | $d_6$ | $\theta_6$ |

where it is assumed that the manipulator has 6 joints. The link displacement $\boldsymbol{T}_i^{i-1}$ for each link is then found from (168), and the end effector displacement is found from $\boldsymbol{T}_n^0 = \boldsymbol{T}_1^0 \dots \boldsymbol{T}_n^{n-1}$.

**Example**

```
% Script for calculation of the end effector displacement
% for the UR5 robot with 6 rotational joints
% from the DH parameters and the joint angles.

dh = zeros(6,4); % Declaration of dh as matrix of dimension 6x4

% Denavit-Hartenberg parameters [a alpha d theta] of UR5 robot
dh(1,:) = [ 0          pi/2   0.0892  0  ];
dh(2,:) = [-0.425      0      0       0  ];
dh(3,:) = [-0.39243    0      0       0  ];
dh(4,:) = [ 0          pi/2   0.109   0  ];
dh(5,:) = [ 0         -pi/2   0.093   0  ];
dh(6,:) = [ 0          0      0.082   0  ];

qh =[0   -pi/2   -pi/2 -pi/2   pi/2  0]'; % Home position

q = qh; % Set joint variables to home position
dhf = dh; % Intermediate variable for calculation
dhf(:,4) = dhf(:,4) + q(:); % Add joint variable to the DH vectors

% Link displacements
T01 =  LinkDH2T (dhf(1,:));
T12 =  LinkDH2T (dhf(2,:));
T23 =  LinkDH2T (dhf(3,:));
T34 =  LinkDH2T (dhf(4,:));
T45 =  LinkDH2T (dhf(5,:));
T56 =  LinkDH2T (dhf(6,:));

% Displaments of intermediate frames
T02 = T01 * T12;
T03 = T02 * T23;
T04 = T03 * T34;
T05 = T04 * T45;

% End effector displacement
T06 = T05 * T56

%%%%%%%%%%%%%%%%%%%%%

function T = LinkDH2T(dh)
% Function for calculating the link displacement from the DH parameters
% dh = [a alpha d theta]
```

```
xv = [1;0;0]; zv=[0;0;1]; r4 = [0 0 0 1];
T =  AngleAxis2T(zv,dh(4)) * [eye(3) dh(3)*zv; r4]...
    * AngleAxis2T(xv,dh(2)) * [eye(3) dh(1)*xv; r4];

%%%%%%%%%%%%%%%%%%%%

function T = AngleAxis2T(r, theta)
% The homogeneous trasformation matrix of a rotation theta about r

r = r/norm(r);
S = [ 0   -r(3)  r(2); r(3) 0    -r(1);-r(2) r(1)  0   ];
R = cos(theta) * eye(3) + sin(theta)*S + (1-cos(theta))*r*r';
T = [R [0;0;0]; ...
    0 0 0 1];
end

%%%%%%%%%%%%%%%%%%%%
```

# 7 Quaternions

## 7.1 Quaternions and Euler parameters

The Euler parameters were introduced by Leonard Euler in 1770. The Euler parameters offer a re-parametrization of the angle-axis parameters with the advantage that the rotation matrix can be written in terms of rational expressions in the Euler parameters. This is an advantage compared to the use of angle-axis parameters where trigonometric expressions are required in the rotation matrix. The Euler parameters describe the rotation matrix with 4 parameters, and the description has no singularities. A further advantage is that the Euler parameters have kinematic differential equations that are well suited for numerical integration used in 3D computer graphics and in control systems. Moreover, Euler parameters are well suited for stability analysis of rotational motion. Kinematics and dynamics related to quaternions in robotics is discussed in [3]. The textbooks [6] and the survey [15] are good sources for the description of rotation in terms of quaternions.

In the following it will be shown that the Euler parameters can be described as unit quaternions in the quaternion formalism introduced by Hamilton in 1843 where a quaternion is a a complex number with one real part and three imaginary parts, while the Euler parameters are described with a scalar and a three-dimensional vector.

The Euler parameters are defined in terms of the angle-axis parameters $\theta$ and $\vec{k}$, and are given by the scalar $\eta$ and the vector $\vec{\epsilon}$ defined by

$$\eta = \cos\frac{\theta}{2}, \qquad \vec{\epsilon} = \vec{k}\sin\frac{\theta}{2} \tag{169}$$

In coordinate form this is written

$$\eta = \cos\frac{\theta}{2}, \qquad \boldsymbol{\epsilon} = \boldsymbol{k}\sin\frac{\theta}{2} \tag{170}$$

We note that

$$\eta^2 + \vec{\epsilon}\cdot\vec{\epsilon} = \eta^2 + \boldsymbol{\epsilon}^{\mathrm{T}}\boldsymbol{\epsilon} = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1 \tag{171}$$

Insertion of the trigonometric identities

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \tag{172}$$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2} \tag{173}$$

into $\boldsymbol{R}_k(\theta) = \cos\theta\boldsymbol{I} + \sin\theta\boldsymbol{k}^\times + (1-\cos\theta)\boldsymbol{k}\boldsymbol{k}^{\mathrm{T}}$ makes it possible to express the rotation matrix $\boldsymbol{R}_{k,\theta}$ in terms of the Euler parameters $\boldsymbol{R}_{k,\theta} = \boldsymbol{R}_e(\eta,\boldsymbol{\epsilon})$, where

$$\boldsymbol{R}_e(\eta,\boldsymbol{\epsilon}) = (\eta^2 - \boldsymbol{\epsilon}^{\mathrm{T}}\boldsymbol{\epsilon})\boldsymbol{I} + 2\eta\boldsymbol{\epsilon}^\times + 2\boldsymbol{\epsilon}\boldsymbol{\epsilon}^{\mathrm{T}} \tag{174}$$

$$= (2\eta^2 - 1)\boldsymbol{I} + 2\eta\boldsymbol{\epsilon} + 2\boldsymbol{\epsilon}\boldsymbol{\epsilon}^{\mathrm{T}} \tag{175}$$

$$= (1 - 2\boldsymbol{\epsilon}^{\mathrm{T}}\boldsymbol{\epsilon})\boldsymbol{I} + 2\eta\boldsymbol{\epsilon}^\times + 2\boldsymbol{\epsilon}\boldsymbol{\epsilon}^{\mathrm{T}} \tag{176}$$

where the different expressions are obtained by using the three alternative expressions in (173).

The alternative form

$$\boldsymbol{R}_e(\eta,\boldsymbol{\epsilon}) = \boldsymbol{I} + 2\eta\boldsymbol{\epsilon}^\times + 2\boldsymbol{\epsilon}^\times\boldsymbol{\epsilon}^\times \tag{177}$$

is found from the expression $\boldsymbol{R}_k(\theta) = \boldsymbol{I} + \sin\theta\boldsymbol{k}^\times + (1-\cos\theta)\boldsymbol{k}^\times\boldsymbol{k}^\times$.

A given rotation will correspond to two sets of Euler parameters $(\eta,\epsilon)$ and $(-\eta,-\epsilon)$ with opposite signs as

$$\boldsymbol{R}_e(-\eta,-\boldsymbol{\epsilon}) = \boldsymbol{R}_e(\eta,\boldsymbol{\epsilon}) \tag{178}$$

The inverse of $\boldsymbol{R}_e(\eta,\epsilon)$ given by

$$\boldsymbol{R}_e(\eta,\boldsymbol{\epsilon})^{\mathrm{T}} = \boldsymbol{R}_e(\eta,-\boldsymbol{\epsilon}) \tag{179}$$

corresponds to the Euler parameters $(\eta,-\epsilon)$.

From (176) and (171) we note that

$$\mathrm{Trace}\boldsymbol{R} = 3(\eta^2 - \boldsymbol{\epsilon}^{\mathrm{T}}\boldsymbol{\epsilon}) + 2\boldsymbol{\epsilon}^{\mathrm{T}}\boldsymbol{\epsilon} = 4\eta^2 - 1 \tag{180}$$

### 7.2 Euler parameters from the rotation matrix

The problem to be solved in this section is how to find the Euler parameters $\eta$, $\boldsymbol{\epsilon}$ when the rotation matrix $\boldsymbol{R} = \{r_{ij}\}$ is given. This is done using Shepperd's method, which was presented in [14].

The rotation matrix is given in terms of the Euler parameters by (176):

$$\boldsymbol{R} = \boldsymbol{R}_e(\eta,\boldsymbol{\epsilon}) = \begin{bmatrix} \eta^2+\epsilon_1^2-\epsilon_2^2-\epsilon_3^2 & 2(\epsilon_1\epsilon_2-\eta\epsilon_3) & 2(\epsilon_1\epsilon_3+\eta\epsilon_2) \\ 2(\epsilon_1\epsilon_2+\eta\epsilon_3) & \eta^2-\epsilon_1^2+\epsilon_2^2-\epsilon_3^2 & 2(\epsilon_2\epsilon_3-\eta\epsilon_1) \\ 2(\epsilon_1\epsilon_3-\eta\epsilon_2) & 2(\epsilon_2\epsilon_3+\eta\epsilon_1) & \eta^2-\epsilon_1^2-\epsilon_2^2+\epsilon_3^2 \end{bmatrix} \tag{181}$$

In addition, $\eta^2+\epsilon_1^2+\epsilon_2^2+\epsilon_3^2 = 1$. The following notation is introduced to simplify the algorithms:

$$\boldsymbol{z} = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} := 2 \begin{bmatrix} \eta \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} \tag{182}$$

$$T := r_{11} + r_{22} + r_{33} = \mathrm{Trace}\boldsymbol{R} \tag{183}$$

and

$$r_{00} := T \tag{184}$$

This gives the symmetric set of equations

$$z_0^2 = 1 + 2r_{00} - T \tag{185}$$
$$z_1^2 = 1 + 2r_{11} - T \tag{186}$$
$$z_2^2 = 1 + 2r_{22} - T \tag{187}$$
$$z_3^2 = 1 + 2r_{33} - T \tag{188}$$

that appear from the diagonal elements of $\boldsymbol{R}$, while the off-diagonal terms give the equations

$$z_0 z_1 = r_{32} - r_{23} \qquad z_2 z_3 = r_{32} + r_{23} \tag{189}$$
$$z_0 z_2 = r_{13} - r_{31} \qquad z_3 z_1 = r_{13} + r_{31} \tag{190}$$
$$z_0 z_3 = r_{21} - r_{12} \qquad z_1 z_2 = r_{21} + r_{12} \tag{191}$$

The algorithm is as follows:

1. Find the largest element in $\{r_{00}, r_{11}, r_{22}, r_{33}\}$. This element is denoted $r_{ii}$.
2. Compute
$$|z_i| = \sqrt{1 + 2r_{ii} - T} \tag{192}$$
3. Determine the sign of $z_i$ from some criterion, like continuity of solution, or $\eta > 0$.
4. Find the remaining $z_j$ from the three equations out of (189–191) that have as the left side $z_j z_i$ for all $j \neq i$. For example, if $z_0$ was found under step 2 and 3, then the remaining $z_j$ are found from
$$z_1 = (r_{32} - r_{23})/z_0 \tag{193}$$
$$z_2 = (r_{13} - r_{31})/z_0 \tag{194}$$
$$z_3 = (r_{21} - r_{12})/z_0 \tag{195}$$
5. Compute $\eta = z_0/2$ and $\epsilon_i = z_i/2$.

Note that this algorithm avoids division by zero as the division is done with the $z_i$ that has the largest absolute value.

```
function [eta, eps] = ShepperdR2q (R)
% Quaternion from rotation matrix using Shepperd's algorithm,
% which is stable, does not lose significant precision and uses only one sqrt.
```

```
% J. Guidance and Control, 1 (1978) 223-224.

    z00 = R(1,1) + R(2,2) + R(3,3); % Trace of R
    z11 = R(1,1) + R(1,1) - z00;
    z22 = R(2,2) + R(2,2) - z00;
    z33 = R(3,3) + R(3,3) - z00;

%Find a large zii to avoid division by zero

    if z00 >= 0.5
        w = sqrt(1.0 + z00);
        wInv = 1.0/w;
        x = (R(3,2) - R(2,3))*wInv;
        y = (R(1,3) - R(3,1))*wInv;
        z = (R(2,1) - R(1,2))*wInv;
    elseif z11 >= 0.5
        x = sqrt(1.0 + z11);
        xInv = 1.0/x;
        w = (R(3,2) - R(2,3))*xInv;
        y = (R(2,1) + R(1,2))*xInv;
        z = (R(3,1) + R(1,3))*xInv;
    elseif z22 >= 0.5
        y = sqrt(1.0 + z22);
        yInv = 1.0/y;
        w = (R(1,3) - R(3,1))*yInv;
        x = (R(2,1) + R(1,2))*yInv;
        z = (R(3,2) + R(2,3))*yInv;
    else
        z = sqrt(1.0 + z33);
        zInv = 1.0/z;
        w = (R(2,1) - R(1,2))*zInv;
        x = (R(3,1) + R(1,3))*zInv;
        y = (R(3,2) + R(2,3))*zInv;
    end

    eta = 0.5*w;
    eps = 0.5*[x; y; z];

end
```

### 7.3 The Euler rotation vector

The Euler rotation vector

$$\boldsymbol{e} = \boldsymbol{k} \sin\theta \tag{196}$$

is defined from the angle-axis parameters $(\boldsymbol{k}, \theta)$ . From (68) it is seen that the rotation matrix $\boldsymbol{R}_{k,\theta}$ and its transpose $\boldsymbol{R}_{k,\theta}^{\mathrm{T}}$ are given by

$$\begin{aligned} \boldsymbol{R}_{k,\theta} &= \boldsymbol{I} + \boldsymbol{e}^{\times} + (1-\cos\theta)\boldsymbol{k}^{\times}\boldsymbol{k}^{\times} & (197) \\ \boldsymbol{R}_{k,\theta}^{\mathrm{T}} &= \boldsymbol{I} - \boldsymbol{e}^{\times} + (1-\cos\theta)\boldsymbol{k}^{\times}\boldsymbol{k}^{\times} & (198) \end{aligned}$$

which implies that

$$\boldsymbol{e}^{\times} = \frac{1}{2}\left(\boldsymbol{R}_{k,\theta} - \boldsymbol{R}_{k,\theta}^{\mathrm{T}}\right) \tag{199}$$

From this we see that if $\boldsymbol{R}_{k,\theta} = \{r_{ij}\}$, then the Euler rotation vector can be found from

$$\boldsymbol{e} = \frac{1}{2}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \tag{200}$$

We note that if $\boldsymbol{R}_{k,\theta} = \boldsymbol{R}_b^a$, then

$$\boldsymbol{e} = \boldsymbol{e}^a = \boldsymbol{e}^b \tag{201}$$

as $\boldsymbol{R}_{k,\theta}\boldsymbol{k} = \boldsymbol{k}$.

In robot control the desired orientation of the robot hand may be specified to be

$$\boldsymbol{R}_d = \begin{bmatrix} \boldsymbol{n}_d & \boldsymbol{s}_d & \boldsymbol{a}_d \end{bmatrix} \in SO(3) \tag{202}$$

Suppose that the actual orientation of the robot hand is

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{n} & \boldsymbol{s} & \boldsymbol{a} \end{bmatrix} \in SO(3) \tag{203}$$

where $\boldsymbol{n}$ is the normal vector, $\boldsymbol{s}$ is the slide vector and $\boldsymbol{a}$ is the approach vector of the hand. Then the deviation of $\boldsymbol{R}$ from $\boldsymbol{R}_d$ is given by the rotation matrix $\widetilde{\boldsymbol{R}} = \{\tilde{r}_{ij}\}$ which is defined by

$$\widetilde{\boldsymbol{R}} := \boldsymbol{R}\boldsymbol{R}_d^{\mathrm{T}} \quad \Rightarrow \quad \boldsymbol{R} = \widetilde{\boldsymbol{R}}\boldsymbol{R}_d \tag{204}$$

The component form of this equation is

$$\tilde{r}_{ij} = n_i n_{dj} + s_i s_{dj} + a_i a_{dj} \tag{205}$$

If an angle-axis parameters of $\widetilde{\boldsymbol{R}}$ are $\left(\widetilde{\boldsymbol{k}}, \tilde{\theta}\right)$ and $\widetilde{\boldsymbol{e}} = \widetilde{\boldsymbol{k}}\sin\tilde{\theta}$ is the associated Euler rotation vector, then (200) gives

$$\begin{aligned} \widetilde{\boldsymbol{e}} &= \frac{1}{2}\begin{bmatrix} \tilde{r}_{32} - \tilde{r}_{23} \\ \tilde{r}_{13} - \tilde{r}_{31} \\ \tilde{r}_{21} - \tilde{r}_{12} \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} n_3 n_{d2} - n_2 n_{d3} \\ n_1 n_{d3} - n_3 n_{d1} \\ n_2 n_{d1} - n_1 n_{d2} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} s_3 s_{d2} - s_2 s_{d3} \\ s_1 s_{d3} - s_3 s_{d1} \\ s_2 s_{d1} - s_1 s_{d2} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} a_3 a_{d2} - a_2 a_{d3} \\ a_1 a_{d3} - a_3 a_{d1} \\ a_2 a_{d1} - a_1 a_{d2} \end{bmatrix} \end{aligned}$$

Using the definition of the vector cross product the Euler rotation vector $\widetilde{\boldsymbol{e}}$ corresponding to the deviation $\widetilde{\boldsymbol{R}}$ can be written

$$\widetilde{\boldsymbol{e}} = \frac{1}{2}(\boldsymbol{n}_d^{\times}\boldsymbol{n} + \boldsymbol{s}_d^{\times}\boldsymbol{s} + \boldsymbol{a}_d^{\times}\boldsymbol{a}) \tag{206}$$

This expression is widely used, although the simpler solution

$$e = \frac{1}{2}\begin{bmatrix} \tilde{r}_{32} - \tilde{r}_{23} \\ \tilde{r}_{13} - \tilde{r}_{31} \\ \tilde{r}_{21} - \tilde{r}_{12} \end{bmatrix} \tag{207}$$

can be obtained from (200).

### 7.4 Quaternions as complex numbers

Hamilton introduced the quaternion on October 16, 1843 [20] as a complex number with one real part and three imaginary parts. This can be written

$$q = q_s + q_1 i + q_2 j + q_3 k$$

where the $q_s$, $q_1$, $q_2$ and $q_3$ are rela scalar, and $i$, $j$ and $k$ are complex units that satisfy the conditions

$$i^2 = j^2 = k^2 = -1$$

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j$$

Multiplication with a scalar $\gamma$ gives

$$\gamma q = \gamma(q_s + q_1 i + q_2 j + q_3 k) = \gamma q_s + \gamma q_1 i + \gamma q_2 j + \gamma q_3 k$$

Let two quaternions be given by $q = q_s + q_1 i + q_2 j + q_3 k$ and $p = p_s + p_1 i + p_2 j + p_3 k$. The the addition is done component-wise as

$$q + p = q_s + p_s + (q_1 + p_1)i + (q_2 + p_2)j + (q_3 + p_3)k$$

while the multiplication of the two quaternions gives

$$qp = (q_s + q_1 i + q_2 j + q_3 k)(p_s + p_1 i + p_2 j + p_3 k) \tag{208}$$
$$= q_s p_s - q_1 p_1 - q_2 p_2 - q_3 p_3 \tag{209}$$
$$+ (q_s p_1 + p_s q_1 + q_2 p_3 - q_3 p_2)i \tag{210}$$
$$+ (q_s p_2 + p_s q_1 + q_3 p_1 - q_1 p_3)j \tag{211}$$
$$+ (q_s p_3 + p_s q_1 + q_1 p_2 - q_2 p_1)k \tag{212}$$

The conjugate of a quaternion is

$$\bar{q} = q_s - iq_1 - jq_2 - kq_3$$

This gives

$$q\bar{q} = q_s^2 + q_1^2 + q_2^2 + q_3^2 \tag{213}$$

The magnitude $\|\boldsymbol{q}\|$ is defined as

$$\|\boldsymbol{q}\|^2 = q\bar{q}$$

### 7.5 Quaternions in the form of a scalar and a vector

A convenient description is to let a quaternion be the sum of a scalar $\alpha$ and a column vector $\boldsymbol{\beta}$, and then impose rules of calculation that are consistent with the rules of calculation for the complex formulation of the quaternions. Then a quaternion is written

$$\boldsymbol{q} = \alpha + \boldsymbol{\beta} \tag{214}$$

where the scalar is the real part and the coefficients of the complex parts are contained in the vector.

The rules of calculation are then as follows. Given two quaternions $\boldsymbol{q}_1 = \alpha_1 + \boldsymbol{\beta}_1$ and $\boldsymbol{q}_2 = \alpha_2 + \boldsymbol{\beta}_2$, addition is given by

$$\boldsymbol{q}_1 + \boldsymbol{q}_2 = \alpha_1 + \alpha_2 + \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2 \tag{215}$$

Multiplication with a scalar $\gamma$ gives

$$\gamma \boldsymbol{q} = \gamma\alpha + \gamma\boldsymbol{\beta} \tag{216}$$

while the quaternion product is written

$$\boldsymbol{q}_1 \circ \boldsymbol{q}_2 = (\alpha_1\alpha_2 - \boldsymbol{\beta}_1^{\mathrm{T}}\boldsymbol{\beta}_2) + (\alpha_1\boldsymbol{\beta}_2 + \alpha_2\boldsymbol{\beta}_1 + \boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2) \tag{217}$$

It is straightforward to verify that this is consistent with the quaternion product (208) in the complex formulation.

The conjugate quaternion is given by

$$\bar{\boldsymbol{q}} = \alpha - \boldsymbol{\beta} \tag{218}$$

so that

$$\boldsymbol{q} \circ \bar{\boldsymbol{q}} = \alpha^2 + \boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\beta} \tag{219}$$

The identity element is $\boldsymbol{q}_{id} = 1$ so that

$$\boldsymbol{q} \circ \boldsymbol{q}_{id} = \boldsymbol{q}_{id} \circ \boldsymbol{q} = \boldsymbol{q} \tag{220}$$

The magnitude $\|\boldsymbol{q}\|$ of a quaternion satisfies

$$\|\boldsymbol{q}\|^2 = \boldsymbol{q} \circ \bar{\boldsymbol{q}} = \alpha^2 + \boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\beta} \tag{221}$$

The inverse $\boldsymbol{q}^{-1}$ of a quaternion $\boldsymbol{q}$ is defined by

$$\boldsymbol{q}^{-1} \circ \boldsymbol{q} = \boldsymbol{q} \circ \boldsymbol{q}^{-1} = \boldsymbol{q}_{id} = 1 \tag{222}$$

It follows that

$$\boldsymbol{q}^{-1} = \frac{\bar{\boldsymbol{q}}}{\boldsymbol{q} \circ \bar{\boldsymbol{q}}} = \frac{\alpha - \boldsymbol{\beta}}{\alpha^2 + \boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\beta}} \tag{223}$$

The conjugate of the quaternion product is

$$\overline{\boldsymbol{q}_1 \circ \boldsymbol{q}_2} = (\alpha_1\alpha_2 - \boldsymbol{\beta}_1^{\mathrm{T}}\boldsymbol{\beta}_2) - (\alpha_1\boldsymbol{\beta}_2 + \alpha_2\boldsymbol{\beta}_1 + \boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2) = \bar{\boldsymbol{q}}_2 \circ \bar{\boldsymbol{q}}_1 \tag{224}$$

The commutator of two quaternions is given by

$$\boldsymbol{q}_1 \circ \boldsymbol{q}_2 - \boldsymbol{q}_2 \circ \boldsymbol{q}_1 = 2\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2 \tag{225}$$

as the cross product term is the only term that changes sign when the order of the factors in the quaternion product changes.

It is noted that the anti-commutator is

$$\boldsymbol{q}_1 \circ \bar{\boldsymbol{q}}_2 + \boldsymbol{q}_2 \circ \bar{\boldsymbol{q}}_1 = 2(\alpha_1\alpha_2 + \boldsymbol{\beta}_1 \cdot \boldsymbol{\beta}_2) \tag{226}$$

### 7.6 Vectors as quaternions

A vector $\boldsymbol{v}$ can be regarded as a quaternion with zero scalar part. The quaternion product of two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ is

$$\boldsymbol{v}_1 \circ \boldsymbol{v}_2 = -\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 + \boldsymbol{v}_1 \times \boldsymbol{v}_2 \tag{227}$$

The conjugate of a vector is $\bar{\boldsymbol{v}} = -\boldsymbol{v}$. The quaternion magnitude of a vector $\boldsymbol{v}$ is consistent with the vector magnitude, which is seen from

$$\|\boldsymbol{v}^2\| = \boldsymbol{v} \circ \bar{\boldsymbol{v}} = -\boldsymbol{v} \circ \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{v} \tag{228}$$

### 7.7 Unit quaternions

A unit quaternion is s quaternion with magnitude 1. A unit quaternion $\boldsymbol{q} = \alpha + \boldsymbol{\beta}$ and satisfies

$$\|\boldsymbol{q}\|^2 = \boldsymbol{q} \circ \bar{\boldsymbol{q}} = \alpha^2 + \boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\beta} = 1 \tag{229}$$

A unit quaternion $\boldsymbol{q} = \alpha + \boldsymbol{\beta}$ can always be described in terms of the Euler parameters $\alpha = \eta$ and $\boldsymbol{\beta} =$ as

$$\boldsymbol{q} = \eta + \boldsymbol{\epsilon} = \cos\frac{\theta}{2} + \boldsymbol{k}\sin\frac{\theta}{2} \tag{230}$$

where $\boldsymbol{k}$ is a unit vector. This is obviously a unit quaternion as $\eta^2 + \boldsymbol{\epsilon}^{\mathrm{T}}\boldsymbol{\epsilon} = 1$.

The sum of two unit quaternions will in general not be a unit quaternion, and the multiplication of a unit quaternion with a scalar will in general not be a unit quaternion.

The identity quaternion $\boldsymbol{q}_{id} = 1$ is a unit quaternion, and the conjugate quaternion $\bar{\boldsymbol{q}} = \eta - \boldsymbol{\epsilon}$ is a unit quaternion. Moreover, since the magnitude is unity, the inverse of a unit quaternion is the conjugate, so that $\boldsymbol{q}^{-1} = \bar{\boldsymbol{q}}$, and

$$\boldsymbol{q} \circ \bar{\boldsymbol{q}} = \bar{\boldsymbol{q}} \circ \boldsymbol{q} = \boldsymbol{q}_{id} = 1 \tag{231}$$

The quaternion product

$$\boldsymbol{q} = \boldsymbol{q}_1 \circ \boldsymbol{q}_2 = \eta_1\eta_2 - \boldsymbol{\epsilon}_1^{\mathrm{T}}\boldsymbol{\epsilon}_2 + \eta_1\boldsymbol{\epsilon}_2 + \eta_2\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_1^{\times}\boldsymbol{\epsilon}_2 \tag{232}$$

of two unit quaternions $\boldsymbol{q}_1 = \eta_1 + \boldsymbol{\epsilon}_1$ and $\boldsymbol{q}_2 = \eta_2 + \boldsymbol{\epsilon}_2$ is a unit quaternion, which is seen from

$$\begin{aligned}
\boldsymbol{q}_1 \circ \boldsymbol{q}_2 =&(\eta_1\eta_2 - \boldsymbol{\epsilon}_1^{\mathrm{T}}\boldsymbol{\epsilon}_2)^2 + (\eta_1\boldsymbol{\epsilon}_2 + \eta_2\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_1^{\times}\boldsymbol{\epsilon}_2)^{\mathrm{T}}(\eta_1\boldsymbol{\epsilon}_2 + \eta_2\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_1^{\times}\boldsymbol{\epsilon}_2)\\
=&\eta_1^2\eta_2^2 - 2\eta_1\eta_2\boldsymbol{\epsilon}_1^{\mathrm{T}}\boldsymbol{\epsilon}_2 + (\boldsymbol{\epsilon}_1^{\mathrm{T}}\boldsymbol{\epsilon}_2)^2 + \eta_1^2\boldsymbol{\epsilon}_2^{\mathrm{T}}\boldsymbol{\epsilon}_2 + 2\eta_1\eta_2\boldsymbol{\epsilon}_1^{\mathrm{T}}\boldsymbol{\epsilon}_2 + \eta_2^2\boldsymbol{\epsilon}_1^{\mathrm{T}}\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_2^{\mathrm{T}}\boldsymbol{\epsilon}_1^{\times}\boldsymbol{\epsilon}_1^{\times}\boldsymbol{\epsilon}_2\\
=&\eta_1^2\eta_2^2 + (\boldsymbol{\epsilon}_1^{\mathrm{T}}\boldsymbol{\epsilon}_2)^2 + \eta_1^2\boldsymbol{\epsilon}_2^{\mathrm{T}}\boldsymbol{\epsilon}_2 + \eta_2^2\boldsymbol{\epsilon}_1^{\mathrm{T}}\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_2^{\mathrm{T}}(\boldsymbol{\epsilon}_1\boldsymbol{\epsilon}_1^{\mathrm{T}} - \boldsymbol{\epsilon}_1^{\mathrm{T}}\boldsymbol{\epsilon}_1)\boldsymbol{\epsilon}_2\\
=&\eta_1^2\eta_2^2 + \eta_1^2\boldsymbol{\epsilon}_2^{\mathrm{T}}\boldsymbol{\epsilon}_2 + \eta_2^2\boldsymbol{\epsilon}_1^{\mathrm{T}}\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2^{\mathrm{T}}\boldsymbol{\epsilon}_2\boldsymbol{\epsilon}_1^{\mathrm{T}}\boldsymbol{\epsilon}_1\\
=&(\eta_1^2 + \boldsymbol{\epsilon}_1^{\mathrm{T}}\boldsymbol{\epsilon}_1)(\eta_2^2 + \boldsymbol{\epsilon}_2^{\mathrm{T}}\boldsymbol{\epsilon}_2)\\
=&1
\end{aligned} \tag{233}$$

It is noted that if the unit quaternion is represented by a 4-dimensional vector, then

$$\|\boldsymbol{q}\|^2 = \begin{bmatrix} \eta \\ \boldsymbol{\epsilon} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \eta \\ \boldsymbol{\epsilon} \end{bmatrix} = \eta^2 + \boldsymbol{\epsilon}^{\mathrm{T}}\boldsymbol{\epsilon} = 1 \tag{234}$$

### 7.8 Rotation of a vector using a quaternion

Consider a rotation matrix $\boldsymbol{R} = \boldsymbol{I} + \sin\theta\boldsymbol{k}^{\times} + (1-\cos\theta)\boldsymbol{k}^{\times}\boldsymbol{k}^{\times}$, which corresponds to a rotation $\theta$ about the unit vector $\boldsymbol{k}$. Let $\boldsymbol{q} = \eta + \boldsymbol{\epsilon} = \cos\frac{\theta}{2} + \boldsymbol{k}\sin\frac{\theta}{2}$ be the unit quaternion defined with the same axis $\boldsymbol{k}$ and angle $\theta$. Then, according to (177), the rotation matrix can be written

$$\boldsymbol{R} = \boldsymbol{I} + 2\eta\boldsymbol{\epsilon}^{\times} + 2\boldsymbol{\epsilon}^{\times}\boldsymbol{\epsilon}^{\times} \tag{235}$$

The rotation of a vector $\boldsymbol{v}$ by the rotation matrix $\boldsymbol{R} \in SO(3)$ is given by $\boldsymbol{R}\boldsymbol{v}$. An important result is that this rotation can be described in terms of the quaternion with the expression

$$\boldsymbol{q} \circ \boldsymbol{v} \circ \bar{\boldsymbol{q}} = \boldsymbol{R}\boldsymbol{v} \tag{236}$$

This is often referred to as a sandwich product. With some patience, this can be verified by the calculation

$$\boldsymbol{q} \circ \boldsymbol{v} \circ \bar{\boldsymbol{q}} =(\eta + \boldsymbol{\epsilon}) \circ \boldsymbol{v} \circ (\eta - \boldsymbol{\epsilon}) \tag{237}$$

$$=(-\boldsymbol{\epsilon}^{\mathrm{T}}\boldsymbol{v} + \eta\boldsymbol{v} + \boldsymbol{\epsilon}^{\times}\boldsymbol{v}) \circ (\eta - \boldsymbol{\epsilon}) \tag{238}$$

$$= -\eta\boldsymbol{\epsilon}^{\mathrm{T}}\boldsymbol{v} + \eta\boldsymbol{v}^{\mathrm{T}}\boldsymbol{\epsilon} + (\boldsymbol{\epsilon}^{\times}\boldsymbol{v})^{\mathrm{T}}\boldsymbol{\epsilon} \tag{239}$$

$$\boldsymbol{\epsilon}\boldsymbol{\epsilon}^{\mathrm{T}}\boldsymbol{v} + \eta^2\boldsymbol{v} + \eta\boldsymbol{\epsilon}^{\times}\boldsymbol{v} + \eta\boldsymbol{\epsilon}^{\times}\boldsymbol{v} + \boldsymbol{\epsilon}^{\times}\boldsymbol{\epsilon}^{\times}\boldsymbol{v} \tag{240}$$

$$=(\boldsymbol{\epsilon}\boldsymbol{\epsilon}^{\mathrm{T}} + \eta^2\boldsymbol{I} + 2\eta\boldsymbol{\epsilon}^{\times} + \boldsymbol{\epsilon}^{\times}\boldsymbol{\epsilon}^{\times})\boldsymbol{v} \tag{241}$$

where it is used that $-(\boldsymbol{\epsilon}^{\times}\boldsymbol{v})^{\times}\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\times}(\boldsymbol{\epsilon}^{\times}\boldsymbol{v}) = \boldsymbol{\epsilon}^{\times}\boldsymbol{\epsilon}^{\times}\boldsymbol{v}$ and $\boldsymbol{\epsilon}^{\mathrm{T}}\boldsymbol{v}\boldsymbol{\epsilon} = \boldsymbol{\epsilon}\boldsymbol{\epsilon}^{\mathrm{T}}\boldsymbol{v}$. The scalar part of the expression is zero since $\boldsymbol{\epsilon}^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}^{\times}\boldsymbol{v}$ is orthogonal to $\boldsymbol{\epsilon}$. The vector part is simplified using $\boldsymbol{\epsilon}^{\times}\boldsymbol{\epsilon}^{\times} = \boldsymbol{\epsilon}\boldsymbol{\epsilon}^{\mathrm{T}} - \boldsymbol{\epsilon}^{\mathrm{T}}\boldsymbol{\epsilon}\boldsymbol{I}$ and $\eta^2 + \boldsymbol{\epsilon}^{\mathrm{T}}\boldsymbol{\epsilon} = 1$, which gives $\boldsymbol{\epsilon}\boldsymbol{\epsilon}^{\mathrm{T}} = \boldsymbol{\epsilon}^{\times}\boldsymbol{\epsilon}^{\times} + \boldsymbol{I} - \eta^2\boldsymbol{I}$. This gives

$$\boldsymbol{q} \circ \boldsymbol{v} \circ \bar{\boldsymbol{q}} =(\boldsymbol{I} + 2\eta\boldsymbol{\epsilon}^{\times} + 2\boldsymbol{\epsilon}^{\times}\boldsymbol{\epsilon}^{\times})\boldsymbol{v} = \boldsymbol{R}\boldsymbol{v} \tag{242}$$

### 7.9 Composite rotations in terms of quaternions

A composite rotation

$$\boldsymbol{R} = \boldsymbol{R}_1\boldsymbol{R}_2 \tag{243}$$

can then be described with the quaternion

$$\boldsymbol{q} = \boldsymbol{q}_1 \circ \boldsymbol{q}_2 \tag{244}$$

where $\boldsymbol{q}_1 = \eta_1 + \boldsymbol{\epsilon}_1$ corresponds to $\boldsymbol{R}_1$ and $\boldsymbol{q}_2 = \eta_2 + \boldsymbol{\epsilon}_2$ corresponds to $\boldsymbol{R}_2$. This is seen from

$$\boldsymbol{q} \circ \boldsymbol{v} \circ \bar{\boldsymbol{q}} = \boldsymbol{q}_1 \circ (\boldsymbol{q}_2 \circ \boldsymbol{v} \circ \bar{\boldsymbol{q}}_2) \circ \bar{\boldsymbol{q}}_1 = \boldsymbol{q}_1 \circ (\boldsymbol{R}_2\boldsymbol{v}) \circ \bar{\boldsymbol{q}}_1 = \boldsymbol{R}_1\boldsymbol{R}_2\boldsymbol{v} \tag{245}$$

Here it is used that $\bar{\boldsymbol{q}} = \overline{\boldsymbol{q}_1 \circ \boldsymbol{q}_2} = \bar{\boldsymbol{q}}_2 \circ \bar{\boldsymbol{q}}_1$.

### 7.10 Deviation in rotation

In this section the description of a deviation in rotation will be described in terms of quaternions. Suppose that a desired rotation from a spatial frame $s$ to a body frame $b$ is represented by the quaternion $\boldsymbol{q}_d = \eta_d + \boldsymbol{\epsilon}_d$, while the actual rotation is given by a quaternion $\boldsymbol{q} = \eta + \boldsymbol{\epsilon}$. The deviation between the two rotations can be related to the body frame by the error quaternion $\boldsymbol{q}_b = \eta_b + \boldsymbol{\epsilon}_b$ according to

$$\boldsymbol{q}_d = \boldsymbol{q} \circ \boldsymbol{q}_b \quad \text{where } \boldsymbol{q}_b = \bar{\boldsymbol{q}} \circ \boldsymbol{q}_d \tag{246}$$

Then

$$\boldsymbol{q}_b = (\eta_d + \boldsymbol{\epsilon}_d) \circ (\eta - \boldsymbol{\epsilon}) = \eta_d\eta + \boldsymbol{\epsilon}_d \cdot \boldsymbol{\epsilon} - \eta_d\boldsymbol{\epsilon} + \eta\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_d \times \boldsymbol{\epsilon} \tag{247}$$

Alternatively, the deviation can be related to spatial frame by the error quaternion $\boldsymbol{q}_s$ according to

$$\boldsymbol{q}_d = \boldsymbol{q}_s \circ \boldsymbol{q} \quad \text{where } \boldsymbol{q}_s = \boldsymbol{q}_d \circ \bar{\boldsymbol{q}} \tag{248}$$

### 7.11 Normalization of unit quaternions

In numerical computations of a unit quaternion it may happen that the computed quaternion $\boldsymbol{p}$ is not a unit quaternion, that is, the condition $\boldsymbol{p} \circ \bar{\boldsymbol{p}} = 1$ does not hold. In that case the computed quaternion can be normalized by

$$\boldsymbol{q} = \frac{\boldsymbol{p}}{\|\boldsymbol{p}\|} \tag{249}$$

where $\|\boldsymbol{p}\| = \sqrt{\boldsymbol{p} \circ \bar{\boldsymbol{p}}}$, which will ensure the condition $\boldsymbol{q} \circ \bar{\boldsymbol{q}} = 1$. This can be used in numerical solution of differential equations involving unit quaternions.

### 7.12 Logarithm of a unit quaternion

Consider the quaternion exponential function defined by

$$\exp(\theta\boldsymbol{k}) = 1 + \theta\boldsymbol{k} + \frac{(\theta\boldsymbol{k})^2}{2!} + \frac{(\theta\boldsymbol{k})^3}{3!} + \ldots \tag{250}$$

where $\boldsymbol{k}^n$ is the quaternion product of $\boldsymbol{k}$ of order $n$. It is noted that $\boldsymbol{k}^2 = \boldsymbol{k} \circ \boldsymbol{k} = -\boldsymbol{k} \cdot \boldsymbol{k} = -1$, while $\boldsymbol{k}^3 = \boldsymbol{k}^2 \circ \boldsymbol{k} = -\boldsymbol{k}$. This gives

$$\exp(\theta\boldsymbol{k}) = 1 + \theta\boldsymbol{k} + \frac{(\theta\boldsymbol{k})^2}{2!} + \frac{(\theta\boldsymbol{k})^3}{3!} + \frac{(\theta\boldsymbol{k})^4}{4!} + \frac{(\theta\boldsymbol{k})^5}{5!} + \ldots \tag{251}$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots\right) + \left(\theta - \frac{\theta^2}{2!} + \frac{\theta^5}{5!} \ldots\right)\boldsymbol{k} \tag{252}$$

$$= \cos\theta + \boldsymbol{k}\sin\theta \tag{253}$$

From this result it is concluded that the unit quaternion can be expressed in terms of the quaternion exponential finction as

$$\boldsymbol{q} = \exp\left(\frac{\theta}{2}\boldsymbol{k}\right) \tag{254}$$

In accordance with this the logarithm of the quaternion is defined by

$$\log(\boldsymbol{q}) = \frac{\theta}{2}\boldsymbol{k} \tag{255}$$

## 8 Kinematic differential equations

### 8.1 The time derivative of a coordinate vector

Consider the vector $\boldsymbol{u}^a = [u_1^a, u_2^a, u_3^a]^{\mathrm{T}}$ given in the coordinates of a frame $a$. The time derivative of the vector is written

$$\dot{\boldsymbol{u}}^a \triangleq \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}^a) = \begin{bmatrix} \dot{u}_1^a \\ \dot{u}_2^a \\ \dot{u}_3^a \end{bmatrix} \tag{256}$$

Note that this is the time derivative of the components of the vector.

The same vector is given by $\boldsymbol{u}^b = [u_1^b, u_2^b, u_3^b]^{\mathrm{T}}$ in another frame $b$. The time derivative of $\boldsymbol{u}^b$ is

$$\dot{\boldsymbol{u}}^b = \begin{bmatrix} \dot{u}_1^b \\ \dot{u}_2^b \\ \dot{u}_3^b \end{bmatrix}$$

Let the rotation matrix from $a$ to $b$ be $\boldsymbol{R}_b^a$. Then the coordinate transformation is given by $\boldsymbol{u}^a = \boldsymbol{R}_b^a\boldsymbol{u}^b$. The time differentiation of the coordinate transformation gives

$$\dot{\boldsymbol{u}}^a = \boldsymbol{R}_b^a\dot{\boldsymbol{u}}^b + \dot{\boldsymbol{R}}_b^a\boldsymbol{u}^b \tag{257}$$

Insertion of $\dot{\boldsymbol{R}}_b^a = \boldsymbol{R}_b^a(\boldsymbol{\omega}_{ab}^b)^\times$ gives the result

$$\dot{\boldsymbol{u}}^a = \boldsymbol{R}_b^a[\dot{\boldsymbol{u}}^b + (\boldsymbol{\omega}_{ab}^b)^\times\boldsymbol{u}^b] \tag{258}$$

**Example 1**

Suppose that the vector $\boldsymbol{u}^b$ is a constant vector, which means that the vector is fixed in frame $b$. Then $\dot{\boldsymbol{u}}^b = \mathbf{0}$, and it follows that

$$\dot{\boldsymbol{u}}^a = \boldsymbol{R}^a_b[(\boldsymbol{\omega}^b_{ab})^\times \boldsymbol{u}^b] = (\boldsymbol{\omega}^a_{ab})^\times \boldsymbol{u}^a \tag{259}$$

□

**Example 2**

Consider the time derivative of the angular velocity vector $\boldsymbol{\omega}^a_{ab}$ of frame $b$ relative to frame $a$ in the coordinates of $a$. Then $\dot{\boldsymbol{\omega}}^a_{ab} = \boldsymbol{R}^a_b[\dot{\boldsymbol{\omega}}^b_{ab} + (\boldsymbol{\omega}^b_{ab})^\times \boldsymbol{\omega}^b_{ab}]$, where $(\boldsymbol{\omega}^b_{ab})^\times \boldsymbol{\omega}^b_{ab} = \mathbf{0}$, which gives

$$\dot{\boldsymbol{\omega}}^a_{ab} = \boldsymbol{R}^a_b \dot{\boldsymbol{\omega}}^b_{ab} \tag{260}$$

□

### 8.2 The time derivative of a coordinate free vector

Let the frame $a$ be defined with orthogonal unit vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$. The coordinate-free form of the vector $\boldsymbol{u}^a$ of the previous section is

$$\vec{u} = u^a_1 \vec{a}_1 + u^a_2 \vec{a}_2 + u^a_3 \vec{a}_3$$

where the vector is described in terms of its components along the orthogonal unit vectors of $a$. The same vector can be given in terms of its components along the orthogonal unit vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3$ of the frame $b$ as

$$\vec{u} = u^b_1 \vec{b}_1 + u^b_2 \vec{b}_2 + u^b_3 \vec{b}_3$$

The time derivative of the vector $\vec{u}$ must be referenced to a coordinate frame to be well defined. To emphasize this the following notation is introduced:

$$\frac{{}^a\mathrm{d}}{\mathrm{d}t}\vec{u} \triangleq \dot{u}^a_1 \vec{a}_1 + \dot{u}^a_2 \vec{a}_2 + \dot{u}^a_3 \vec{a}_3 \tag{261}$$

This is said to be the time derivative of the vector $\vec{u}$ in frame $a$. It is seen that the derivation is done by differentiating the coordinates $u^a_i$, while keeping the orthogonal vectors. In the same way the time derivative of $\vec{u}$ in $b$ is given by

$$\frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{u} = \dot{u}^b_1 \vec{b}_1 + \dot{u}^b_2 \vec{b}_2 + \dot{u}^b_3 \vec{b}_3 \tag{262}$$

From (258) it follows that

$$\frac{{}^a\mathrm{d}}{\mathrm{d}t}\vec{u} = \frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{u} + \vec{\omega}_{ab} \times \vec{u} \tag{263}$$

**Example 1**

Suppose that the vector $\vec{u}$ is fixed in $b$. Then the time derivative of $\vec{u}$ in $b$ is zero, and

$$\frac{{}^a\mathrm{d}}{\mathrm{d}t}\vec{u} = \vec{\omega}_{ab} \times \vec{u} \tag{264}$$

□

**Example 2**

Consider the time derivative of the angular velocity vector $\vec{\omega}_{ab}$ of frame $b$ relative to frame $a$. Then $\frac{{}^a\mathrm{d}}{\mathrm{d}t}\vec{\omega}_{ab} = \frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{\omega}_{ab} + \vec{\omega}_{ab} \times \vec{\omega}_{ab}$, and since $\vec{\omega}_{ab} \times \vec{\omega}_{ab} = \vec{0}$, it follows that

$$\frac{{}^a\mathrm{d}}{\mathrm{d}t}\vec{\omega}_{ab} = \frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{\omega}_{ab} \tag{265}$$

□

## 8.3 The velocity vector

Consider a point with position vector $\boldsymbol{p}^n = [x, y, z]^{\mathrm{T}}$, where $\boldsymbol{p}^n$ is given in the coordinates of of a fixed frame $n$. The frame $n$ will typically be a Newtonian or inertial frame, which is a frame where Newton's law applies.

Then the velocity $\boldsymbol{v}^n$ of the point will be the time derivative

$$\boldsymbol{v}^n = \dot{\boldsymbol{p}}^n = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \tag{266}$$

of the position vector in the $n$ frame.

In the coordinate-free description the position vector is $\vec{p} = x\vec{n}_1 + y\vec{n}_2 + z\vec{n}_3$, and the velocity is

$$\vec{v} = \frac{{}^n\mathrm{d}}{\mathrm{d}t}\vec{p} = \dot{x}\vec{n}_1 + \dot{y}\vec{n}_2 + \dot{z}\vec{n}_3 \tag{267}$$

where $\vec{n}_1, \vec{n}_2, \vec{n}_3$ are the orthogonal unit vectors of frame $n$.

It is noted that the acceleration of the point is $\boldsymbol{a}^n = \ddot{\boldsymbol{p}}^n = [\ddot{x}, \ddot{y}, \ddot{z}]^{\mathrm{T}}$, which is written

$$\vec{a} = \frac{{}^n\mathrm{d}^2}{\mathrm{d}t^2}\vec{p} = \frac{{}^n\mathrm{d}}{\mathrm{d}t}\vec{v} = \ddot{x}\vec{n}_1 + \ddot{y}\vec{n}_2 + \ddot{z}\vec{n}_3 \tag{268}$$

in the coordinate-free form.

## 8.4 The velocity of a link attached with a rotary joint

Consider a single link which is a rigid body with a body-fixed frame $b$. The link is fixed to a fixed frame $n$ with a rotary joint. The rotation angle is $\theta$, and the joint axis is through the origin of $n$. The origin of the $b$ frame has position $\vec{r}_b$. Note that since the frame $b$ is fixed in the rigid body, the vector $\vec{r}_b$ will be constant in frame $b$, which means that $\frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{r}_b = \vec{0}$. The velocity of the origin of frame $b$ is then

$$\vec{v}_b = \frac{{}^n\mathrm{d}}{\mathrm{d}t}\vec{r}_b = \vec{\omega}_{nb} \times \vec{r}_b \tag{269}$$

### 8.5 The velocity of the end effector of a serial link manipulator

Consider a manipulator which is a serial connection of 6 links connected with rotary joints. A frame is fixed in each link so that frame $i$ is fixed in link $i$. The position of the end effector is then

$$\vec{p}_6 = \vec{p}_{01} + \vec{p}_{12} + \ldots + \vec{p}_{56} \tag{270}$$

where $\boldsymbol{p}_i$ is the position of the origin of frame $i$, and $\vec{p}_{i,i-1} = \vec{p}_i - \vec{p}_{i-1}$ is the position vector from the origin of frame $i-1$ to the origin of frame $i$. It follows that the vector $\vec{p}_{i,i-1}$ will be fixed in frame $i$, so that

$$\frac{{}^0\mathrm{d}}{\mathrm{d}t}\vec{p}_{i,i-1} = \frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{p}_{i,i-1} + \vec{\omega}_{0i} \times \vec{p}_{i,i-1} = \vec{\omega}_{0i} \times \vec{p}_{i,i-1} \tag{271}$$

The velocity $\vec{v}_6 = \frac{{}^0\mathrm{d}}{\mathrm{d}t}\vec{p}_6$ of frame 6 is then

$$\vec{v}_6 = \vec{\omega}_{01} \times \vec{p}_{01} + \vec{\omega}_{02} \times \vec{p}_{12} + \ldots + \vec{\omega}_{06} \times \vec{p}_{56} \tag{272}$$

It is convenient to rewrite this in the form

$$\vec{v}_6 = \vec{\omega}_{01} \times \vec{p}_{06} + \vec{\omega}_{12} \times \vec{p}_{26} + \ldots + \vec{\omega}_{56} \times \vec{p}_{56} \tag{273}$$

where $\vec{p}_{i6} = \vec{p}_6 - \vec{p}_i$ is the position vector from frame $i$ to frame 6. In the Denavit-Hartenberg convention the angular velocity vector between two consecutive frames is

$$\vec{\omega}_{i-1,i} = \dot{\theta}_i \vec{z}_i \tag{274}$$

where $\vec{z}_i$ is the unit vector of the $z$ axis of frame $i$. This gives

$$\vec{v}_6 = \vec{z}_0 \times \vec{p}_{06}\dot{\theta}_1 + \vec{z}_1 \times \vec{p}_{16}\dot{\theta}_2 + \ldots + \vec{z}_5 \times \vec{p}_{56}\dot{\theta}_6 \tag{275}$$

It is noted that the angular velocity of frame 6 will be

$$\vec{\omega}_6 = \vec{z}_0\dot{\theta}_1 + \vec{z}_1\dot{\theta}_2 + \ldots + \vec{z}_5\dot{\theta}_6 \tag{276}$$

Moreover, it is noted that if joint $i$ is prismatic, then the contribution to the velocity will instead be $\vec{z}_i\dot{d}_i$, and the contribution to the angular velocity will be zero.

### 8.6 The angular velocity vector

In this section the time derivative of the rotation matrix will be presented and discussed. It was shown above that the time derivative of a position vector in a fixed frame is the velocity vector. In the case of a rotation matrix the situation is somewhat more complicated. The derivation is based on the observation that a rotation matrix $\boldsymbol{R}^a_b$ from frame $a$ to frame $b$ will satisfy $\boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{I}$. The time derivative of this equation gives

$$\dot{\boldsymbol{R}}^a_b(\boldsymbol{R}^a_b)^{\mathrm{T}} + \boldsymbol{R}^a_b\dot{(\boldsymbol{R}^a_b)^{\mathrm{T}}} = \boldsymbol{0} \tag{277}$$

The matrix $\boldsymbol{S} = \dot{\boldsymbol{R}}_b^a(\boldsymbol{R}_b^a)^{\mathrm{T}}$ with transpose $\boldsymbol{S}^{\mathrm{T}} = \boldsymbol{R}_b^a\dot{\boldsymbol{R}}^{\mathrm{T}}$ will therefore satisfy $\boldsymbol{S} + \boldsymbol{S}^{\mathrm{T}} = \boldsymbol{0}$, which means that $\boldsymbol{S}$ is skew symmetric. A skew symmetric matrix of dimension $3 \times 3$ can always be written

$$\boldsymbol{S} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{278}$$

Define the column vector $\boldsymbol{\omega}_{ab}^a = [\omega_x, \omega_y, \omega_z]^{\mathrm{T}}$. Then $\boldsymbol{S}$ is the skew symmetric form of the vector $\boldsymbol{\omega}_{ab}^a$, which is written $\boldsymbol{S} = (\boldsymbol{\omega}_{ab}^a)^\times$. From $\boldsymbol{S} = \dot{\boldsymbol{R}}_b^a(\boldsymbol{R}_b^a)^{\mathrm{T}}$ is follows that

$$\dot{\boldsymbol{R}}_b^a = (\boldsymbol{\omega}_{ab}^a)^\times \boldsymbol{R}_b^a, \quad (\boldsymbol{\omega}_{ab}^a)^\times = \dot{\boldsymbol{R}}_b^a(\boldsymbol{R}_b^a)^{\mathrm{T}} \tag{279}$$

The vector $\boldsymbol{\omega}_{ab}^a$ is the angular velocity vector of frame $b$ relative to frame $a$ in the coordinates of the $a$ frame.

The coordinate transformation of the angular velocity vector from $b$ to $a$ is given by $\boldsymbol{\omega}_{ab}^a = \boldsymbol{R}_b^a\boldsymbol{\omega}_{ab}^b$, while the skew symmetric form is transformed according to $(\boldsymbol{\omega}_{ab}^a)^\times = \boldsymbol{R}_b^a(\boldsymbol{\omega}_{ab}^b)^\times(\boldsymbol{R}_b^a)^{\mathrm{T}}$. This gives

$$\dot{\boldsymbol{R}}_b^a = \boldsymbol{R}_b^a(\boldsymbol{\omega}_{ab}^b)^\times, \quad (\boldsymbol{\omega}_{ab}^b)^\times = (\boldsymbol{R}_b^a)^{\mathrm{T}}\dot{\boldsymbol{R}}_b^a \tag{280}$$

### 8.7 Kinematic differential equation for a simple rotation

The angular velocity for a simple rotation is found from

$$\boldsymbol{R}_b^a = \boldsymbol{I} + \sin\theta(\boldsymbol{k}^a)^\times + (1 - \cos\theta)(\boldsymbol{k}^a)^\times(\boldsymbol{k}^a)^\times \tag{281}$$

which has time derivative

$$\dot{\boldsymbol{R}}_b^a = \dot{\theta}\left[\cos\theta(\boldsymbol{k}^a)^\times + \sin\theta(\boldsymbol{k}^a)^\times(\boldsymbol{k}^a)^\times\right] \tag{282}$$

The angular velocity is found after some calculation, where it is used that $[(\boldsymbol{k}^a)^\times]^3 = -(\boldsymbol{k}^a)^\times$, to be

$$(\boldsymbol{\omega}_{ab}^a)^\times = \dot{\boldsymbol{R}}_b^a(\boldsymbol{R}_b^a)^{\mathrm{T}} = \dot{\theta}(\boldsymbol{k}^a)^\times \tag{283}$$

which gives the result

$$\boldsymbol{\omega}_{ab}^a = \dot{\theta}\boldsymbol{k}^a \tag{284}$$

This shows that the angular velocity of a simple rotation is along the constant axis of rotation, and that the magnitude of the angular velocity is the rate $\dot{\theta}$ of the rotation angle $\theta$.

### 8.8 Kinematic differential equation for a simple rotation about a coordinate axis

The time derivative of a rotation matrix from frame $a$ to frame $b$ is $\dot{\boldsymbol{R}}_b^a = (\boldsymbol{\omega}_{ab}^a)^\times \boldsymbol{R}_b^a$ where $\boldsymbol{\omega}_{ab}^a$ is the angular velocity of frame $b$ with reference to frame $a$ in the coordinates of frame $a$. In the case of simple rotations about the coordinate axis this gives an interesting result. First, consider the time derivative of $\boldsymbol{R}_x(\phi)$ about the $x_a$ axis. Then it is easily verified that the time derivative satisfies

$$\dot{\boldsymbol{R}}_x(\phi) = (\dot{\phi}\boldsymbol{x})^\times \boldsymbol{R}_x(\phi) \tag{285}$$

where $\boldsymbol{x} = [1, 0, 0]^{\mathrm{T}}$ is the coordinate vector of the $x$ axis in frame 1. This means that the angular velocity is $\boldsymbol{\omega}_{12}^1 = \dot{\phi}\boldsymbol{x}$.

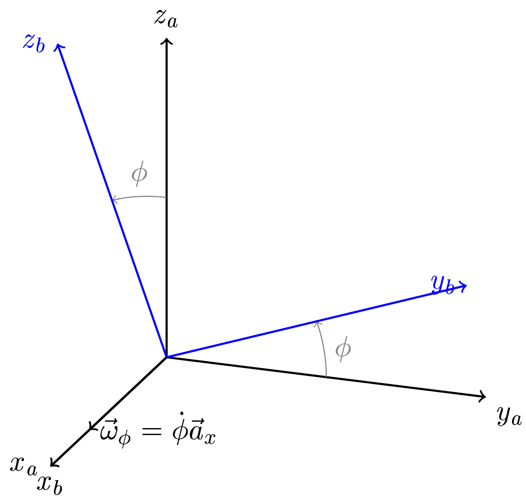



Figure 10: The angular velocity of a simple rotation by an angle $\phi$ about the $x$ axis is $\vec{\omega}_\phi = \dot{\phi}\vec{a}_x$, where $\vec{a}_x$ is the unit vector along the $x_a$ axis.

In the same way it is found that for a simple rotation about the $y_a$ axis, the time derivative of the rotation matrix is

$$\dot{\boldsymbol{R}}_y(\theta) = (\dot{\theta}\boldsymbol{y})^\times \boldsymbol{R}_y(\theta) \tag{286}$$

where $\boldsymbol{y} = [0, 1, 0]^\mathrm{T}$ is the coordinate vector of the $y_a$ axis in frame $a$, which corresponds to an angular velocity $\boldsymbol{\omega}_{ab}^a = \dot{\theta}\boldsymbol{y}$.

Finally, for a simple rotation about the $z$ axis of frame $a$, the time derivative of the rotation matrix is

$$\dot{\boldsymbol{R}}_z(\phi) = (\dot{\psi}\boldsymbol{z})^\times \boldsymbol{R}_z(\psi) \tag{287}$$

where $\boldsymbol{z} = [0, 0, 1]^\mathrm{T}$ is the coordinate vector of the $z_a$ axis in frame $a$, which gives the angular velocity $\boldsymbol{\omega}_{ab}^a = \dot{\psi}\boldsymbol{z}$.

### 8.9 Kinematic differential equations for the roll-pitch-yaw Euler angles

The time derivative of the rotation matrix for the roll-pitch-yaw angles is

$$\dot{\boldsymbol{R}}_{ZYX} = \boldsymbol{\omega}^\times \boldsymbol{R}_{ZYX} \tag{288}$$

where $\boldsymbol{R}_{ZYX} = \boldsymbol{R}_2^1$ is the rotation from frame 1 to frame 2 and $\boldsymbol{\omega} = \boldsymbol{\omega}_{12}^1$ is the angular velocity of frame 2 relative to frame 1 in the coordinates of frame 1. The time derivative can also be developed from the simple rotations $\boldsymbol{R}_z(\psi)$, $\boldsymbol{R}_y(\theta)$ and $\boldsymbol{R}_z(\phi)$ to be

$$\begin{aligned}
\dot{\boldsymbol{R}}_{ZYX} =& \dot{\boldsymbol{R}}_z(\psi)\boldsymbol{R}_y(\theta)\boldsymbol{R}_x(\phi) + \boldsymbol{R}_z(\psi)\dot{\boldsymbol{R}}_y(\theta)\boldsymbol{R}_x(\phi) + \boldsymbol{R}_z(\psi)\boldsymbol{R}_y(\theta)\dot{\boldsymbol{R}}_x(\phi) \\
=& (\dot{\psi}\boldsymbol{z})^\times \boldsymbol{R}_z(\psi)\boldsymbol{R}_y(\theta)\boldsymbol{R}_x(\phi) + \boldsymbol{R}_z(\psi)(\dot{\theta}\boldsymbol{y})^\times \boldsymbol{R}_y(\theta)\boldsymbol{R}_x(\phi) + \boldsymbol{R}_z(\psi)\boldsymbol{R}_y(\theta)(\dot{\phi}\boldsymbol{x})^\times \boldsymbol{R}_x(\phi) \\
=& [(\dot{\psi}\boldsymbol{z})^\times + \boldsymbol{R}_z(\psi)(\dot{\theta}\boldsymbol{y})^\times \boldsymbol{R}_z(\psi)^\mathrm{T} + \boldsymbol{R}_z(\psi)\boldsymbol{R}_y(\theta)(\dot{\phi}\boldsymbol{x})^\times \boldsymbol{R}_y(\theta)^\mathrm{T}\boldsymbol{R}_z(\psi)^\mathrm{T}]\boldsymbol{R}_{ZYX} \\
=& [(\dot{\psi}\boldsymbol{z})^\times + [\boldsymbol{R}_z(\psi)\dot{\theta}\boldsymbol{y}]^\times + [\boldsymbol{R}_z(\psi)\boldsymbol{R}_y(\theta)\dot{\phi}\boldsymbol{x}]^\times]\boldsymbol{R}_{ZYX}
\end{aligned} \tag{289}$$

This shows that the angular velocity is

$$\boldsymbol{\omega} = \dot{\psi}\boldsymbol{z} + \dot{\theta}\boldsymbol{R}_z(\psi)\boldsymbol{y} + \dot{\phi}\boldsymbol{R}_z(\psi)\boldsymbol{R}_y(\theta)\boldsymbol{x} \tag{290}$$

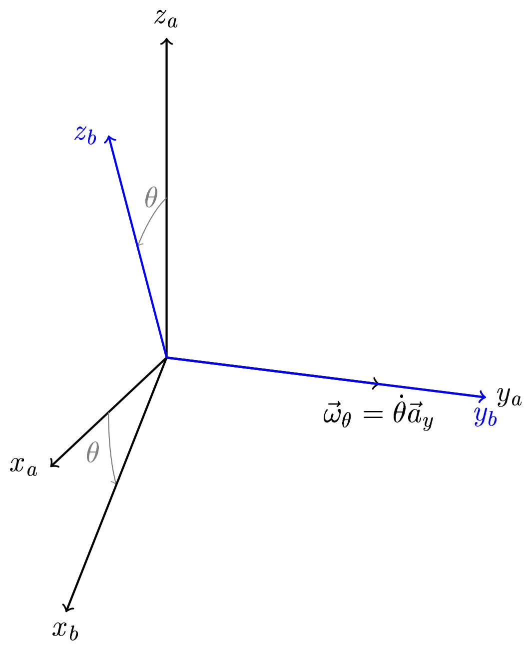



Figure 11: The angular velocity of a simple rotation by an angle $\theta$ about the $x$ axis is $\vec{\omega}_\theta = \dot{\theta}\vec{a}_y$, where $\vec{a}_y$ is the unit vector along the $y_a$ axis.

It is noted that $\boldsymbol{R}_z(\psi)\boldsymbol{y}$ is the second column of the matrix $\boldsymbol{R}_z(\psi)$, while $\boldsymbol{R}_z(\psi)\boldsymbol{R}_y(\theta)\boldsymbol{x}$ can be found efficiently by extracting the first column of $\boldsymbol{R}_y(\theta)$ and then multiplying with $\boldsymbol{R}_z(\psi)$. This gives the expression

$$\boldsymbol{\omega} = \dot{\psi}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \dot{\theta}\begin{bmatrix} -\sin\psi \\ \cos\psi \\ 0 \end{bmatrix} + \dot{\phi}\begin{bmatrix} \cos\psi\cos\theta \\ \sin\psi\cos\theta \\ \sin\theta \end{bmatrix} \tag{291}$$

where the column vectors are the coordinate vectors of the rotation axes in frame 1.

This can also be written

$$\boldsymbol{\omega} = \boldsymbol{E}\dot{\boldsymbol{\phi}} = \underbrace{\begin{bmatrix} \cos\psi\cos\theta & -\sin\psi & 0 \\ \sin\psi\cos\theta & \cos\psi & 0 \\ -\sin\theta & 0 & 1 \end{bmatrix}}_{=\boldsymbol{E}} \underbrace{\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}}_{=\dot{\boldsymbol{\phi}}} \tag{292}$$

where $\boldsymbol{\phi} = [\phi, \theta, \psi]^{\mathrm{T}}$. The inverse relation is

$$\dot{\boldsymbol{\phi}} = \boldsymbol{E}(\boldsymbol{\phi})^{-1}\boldsymbol{\omega} \tag{293}$$

where the inverse of the matrix $\boldsymbol{E}$ is

$$\boldsymbol{E}^{-1} = \frac{1}{\cos\theta}\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi\cos\theta & \cos\psi\cos\theta & 0 \\ \cos\psi\sin\theta & \sin\psi\sin\theta & \cos\theta \end{bmatrix} \tag{294}$$

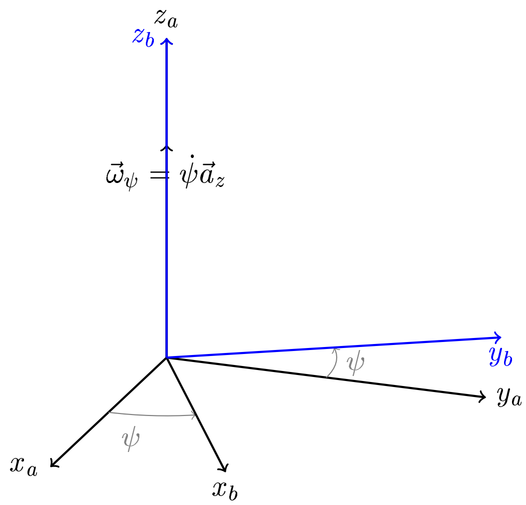



Figure 12: The angular velocity of a simple rotation by an angle $\psi$ about the $z_a$ axis is $\vec{\omega}_\psi = \dot{\psi}\vec{a}_z$, where $\vec{a}_x$ is the unit vector along the $x_a$ axis.

Note that $\boldsymbol{E}$ is singular for $\cos\theta = 0$, and the the inverse is undefined in this case. The geometric interpretation of this singularity will be investigated in detail in the following.

**Example**

The angular velocity can be expressed in the coordinates of frame 2 as $\boldsymbol{\omega}_{12}^2 = \boldsymbol{R}_1^2\boldsymbol{\omega}_{12}^1$. Here $\boldsymbol{R}_1^2 = \left(\boldsymbol{R}_2^1\right)^{\mathrm{T}} = \boldsymbol{R}_x(\phi)^{\mathrm{T}}\boldsymbol{R}_y(\theta)^{\mathrm{T}}\boldsymbol{R}_z(\psi)^{\mathrm{T}}$ which gives

$$\boldsymbol{\omega}_{12}^2 = \dot{\psi}\boldsymbol{R}_x(\phi)^{\mathrm{T}}\boldsymbol{R}_y(\theta)^{\mathrm{T}}\boldsymbol{z} + \dot{\theta}\boldsymbol{R}_x(\phi)^{\mathrm{T}}\boldsymbol{y} + \dot{\phi}\boldsymbol{x} \tag{295}$$

$$= \dot{\psi}\begin{bmatrix} \sin\theta \\ \sin\phi\cos\theta \\ \cos\phi\cos\theta \end{bmatrix} + \dot{\theta}\begin{bmatrix} 0 \\ \cos\psi \\ -\sin\phi \end{bmatrix} + \dot{\phi}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{296}$$

where the columns vectors are the rotation axes in frame 2. This can be written

$$\boldsymbol{\omega}_{12}^2 = \begin{bmatrix} 1 & 0 & \sin\theta \\ 0 & \cos\psi & \sin\phi\cos\theta \\ 0 & -\sin\psi & \cos\phi\cos\theta \end{bmatrix}\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{297}$$

### 8.10 Kinematic differential equations for the ZYZ Euler angles

It was found for the roll-pitch-yaw Euler angles that the angular velocity due a rotational motion described in terms of the time derivatives $\dot{\psi}$, $\dot{\theta}$ and $\dot{\phi}$ of the Euler angles can be found as the sum of the angular velocity due to each of the Euler angles. Moreover, when the rotation is the rotation $\boldsymbol{R}_2^1 = \boldsymbol{R}_{ZYZ}$ from frame 1 to 2, the angular velocity due to $\dot{\psi}$ is an angular velocity of magnitude $\dot{\psi}$ along the rotation angle of $\psi$, which is the $z$ axis of frame 1. The angular velocity due to $\dot{\theta}$ is an angular velocity of magnitude $\dot{\theta}$ along the rotation axis of $\theta$, which is the $y$ axis of the intermediate frame $1'$. Finally, the angular velocity due to $\dot{\phi}$ is of magnitude $\dot{\phi}$ along the $z$ xis of the second intermediate frame $1''$.

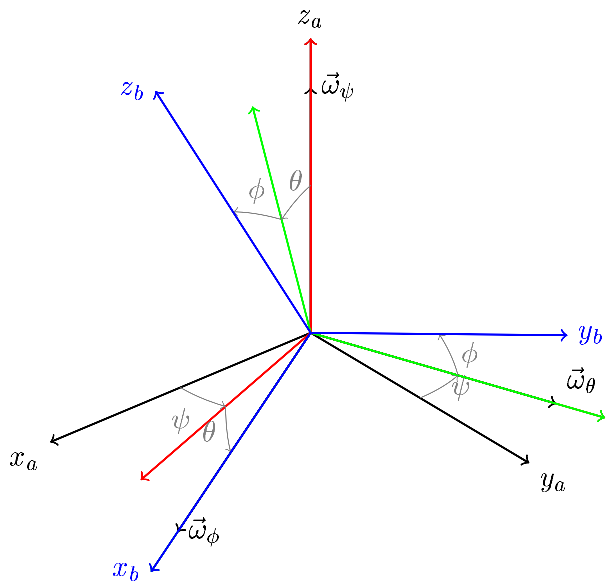



Figure 13: The angular velocity $\vec{\omega}_{ab}$ of frame $a$ relative to frame $b$ is the sum of the angular velocities due to the time derivatives $\dot{\psi}$, $\dot{\theta}$ and $\dot{\phi}$ of the Euler angles. The angular velocities are along the rotated rotation axes. The resulting angular velocity is $\vec{\omega}_{ab} = \vec{\omega}_\psi + \vec{\omega}_\theta + \vec{\omega}_\phi$.

Therefore angular velocity is

$$\boldsymbol{\omega} = \dot{\psi}\boldsymbol{z} + \dot{\theta}\boldsymbol{R}_z(\psi)\boldsymbol{y} + \dot{\phi}\boldsymbol{R}_z(\psi)\boldsymbol{R}_y(\theta)\boldsymbol{z} \tag{298}$$

which gives

$$\boldsymbol{\omega} = \dot{\psi}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \dot{\theta}\begin{bmatrix} -\sin\psi \\ \cos\psi \\ 0 \end{bmatrix} + \dot{\phi}\begin{bmatrix} \cos\psi\sin\theta \\ \sin\psi\sin\theta \\ \cos\theta \end{bmatrix} \tag{299}$$

and $\boldsymbol{\omega} = \boldsymbol{F}\dot{\boldsymbol{\phi}}$ where

$$\boldsymbol{F} = \begin{bmatrix} \cos\psi\sin\theta & -\sin\psi & 0 \\ \sin\psi\sin\theta & \cos\psi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \tag{300}$$

The inverse relation is

$$\dot{\boldsymbol{\phi}} = \boldsymbol{F}^{-1}\boldsymbol{\omega} = \frac{1}{\sin\theta}\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi\sin\theta & \cos\psi\sin\theta & 0 \\ -\cos\psi\cos\theta & -\sin\psi\cos\theta & \sin\theta \end{bmatrix}\boldsymbol{\omega} \tag{301}$$

which has a singularity when $\sin\theta = 0$.

### 8.11 Singularities of the kinematic differential equations for roll-pitch-yaw angles

Consider the kinematic differential equations for the roll-pitch-yaw Euler angles for $\theta = -\pi/2$. Then $\cos\theta = 0$ and the kinematic differential equations are singular in the sense that the matrix

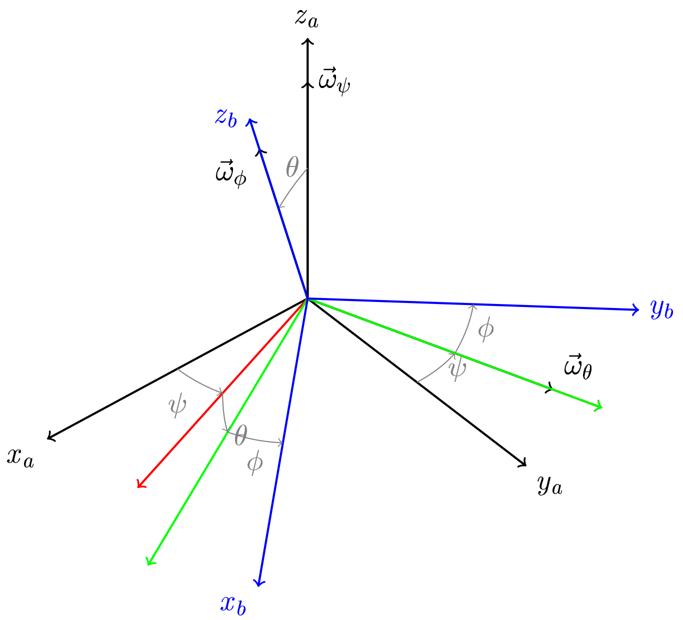



Figure 14: The angular velocity $\vec{\omega}_{ab}$ of frame $a$ relative to frame $b$ is the sum of the angular velocity $\vec{\omega}_\psi$ along the $z_a$ axis due to $\dot{\psi}$, plus the angular velocity $\vec{\omega}_\theta$ along the rotated $y_a'$ axis due to $\dot{\theta}$, plus the angular velocity $\vec{\omega}_\phi$ along the rotated $z_a''$ axis due to $\dot{\phi}$. The resulting angular velocity is $\vec{\omega}_{ab} = \vec{\omega}_\psi + \vec{\omega}_\theta + \vec{\omega}_\phi$.

$\boldsymbol{E}$ is singular. To make the analysis simple it is assumed that $\psi = 0$. Then the kinematic differential equation $\boldsymbol{\omega} = \boldsymbol{E}\dot{\boldsymbol{\phi}}$ can be written

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{302}$$

It is seen that $\omega_x = 0$, $\omega_y = \dot{\theta}$ and $\omega_z = \dot{\psi} + \dot{\phi}$. This means that in this configuration there can be no angular velocity about the $x$ axis. The reason for this is that the $x''$ axis of the rotation $\phi$ is aligned with the $z$ axis of the rotation $\psi$. Note that if $\dot{\psi} = -\dot{\phi}$, then there will be no angular velocity due to $\dot{\psi}$ and $\dot{\phi}$.

The singular value decomposition $\boldsymbol{E} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{T}}$ has the factors

$$\boldsymbol{U} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} 1.4142 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{V} = \begin{bmatrix} -0.7071 & 0 & -0.7071 \\ 0 & 1 & 0 \\ -0.7071 & 0 & 0.7071 \end{bmatrix} \tag{303}$$

The last singular value is $\sigma_3 = 0$, which shows that the matrix $\boldsymbol{E}$ is singular. The following observations can be made from this fact: The corresponding output vector $\boldsymbol{u}_3 = [-1, 0, 0]^{\mathrm{T}}$ is not in the range space of $\boldsymbol{E}$, which means that the output $\boldsymbol{\omega}$ cannot have any component along $\boldsymbol{u}_3$. The corresponding input vector $\boldsymbol{v}_3 = [-0.7071, 0, 0.7071]^{\mathrm{T}}$ is in the nullspace of $\boldsymbol{E}$, which means that an input along $\boldsymbol{v}_3$, that is, and input with $\dot{\psi} = -\dot{\phi}$, will not give any output $\boldsymbol{\omega}$ as remarked above.

Close to the singularity given by $\cos\theta = 0$ the matrix $\boldsymbol{E}$ will be invertible. Consider the configuration $\psi = 0$ and $\theta = -\pi/2 + 0.02$, where $\theta$ is 0.02 rad or about 1 degree away from the singularity. The

$$\boldsymbol{E} = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 1 & 0 \\ 0.998 & 0 & 1 \end{bmatrix} \tag{304}$$

and the singular value decomposition has the factors

$$\boldsymbol{U} = \begin{bmatrix} -0.01 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0.01 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} 1.4141 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.0141 \end{bmatrix}, \quad \boldsymbol{V} = \begin{bmatrix} -0.7071 & 0 & -0.7071 \\ 0 & 1 & 0 \\ -0.7071 & 0 & 0.7071 \end{bmatrix} \tag{305}$$

It is noted that the smallest singular value is $\sigma_3 = 0.0141$, which is not zero, so the matrix is invertible, still the small value of $\sigma_3$ makes the inverse mapping $\boldsymbol{\phi} = \boldsymbol{E}^{-1}\boldsymbol{\omega}$ very sensitive as the amplification in direction defined by $\boldsymbol{u}_3$ and $\boldsymbol{v}_3$ will be $1/\sigma_3 = 70.711$. This is seen by considering the inverse of $\boldsymbol{E}$, which is $\boldsymbol{E}^{-1} = \boldsymbol{V}\boldsymbol{\Sigma}^{-1}\boldsymbol{U}^{\mathrm{T}}$, and the inverse mapping from $\boldsymbol{\omega}$ to $\boldsymbol{\phi}$ can be written in the form

$$\boldsymbol{\phi} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} (\omega_z + 0.01\omega_x) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \omega_y + 70.711 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} (\omega_x - 0.01\omega_z) \tag{306}$$

Here it is seen that if an angular velocity $\omega_x$ of, say, unity magnitude is commanded in the $x$ direction, then there will be rotations of magnitude 70.711 in $\dot{\phi}$ and $-\dot{\psi}$. This shows that the mapping $\boldsymbol{\phi} = \boldsymbol{E}^{-1}\boldsymbol{\omega}$ is very sensitive close to a singularity.

The kinematic differential equations can be written in terms of the singular value decomposition as

$$\dot{\boldsymbol{\phi}} = \sigma_1 \left(\boldsymbol{v}_1^{\mathrm{T}}\boldsymbol{\omega}\right) \boldsymbol{u}_1 + \sigma_2 \left(\boldsymbol{v}_2^{\mathrm{T}}\boldsymbol{\omega}\right) \boldsymbol{u}_2 + \sigma_3 \left(\boldsymbol{v}_3^{\mathrm{T}}\boldsymbol{\omega}\right) \boldsymbol{u}_3 \tag{307}$$

It is seen that if $\sigma_3 = 0$ then there can be no component of $\dot{\boldsymbol{\phi}}$ along the output vector $\boldsymbol{u}_3$, and components of $\boldsymbol{\omega}$ along the input vector $\boldsymbol{v}_3$ will not result in any output $\dot{\boldsymbol{\phi}}$.

The corresponding inverse relation is

$$\boldsymbol{\omega} = \frac{1}{\sigma_1} \left(\boldsymbol{u}_1^{\mathrm{T}}\dot{\boldsymbol{\phi}}\right) \boldsymbol{v}_1 + \frac{1}{\sigma_2} \left(\boldsymbol{u}_2^{\mathrm{T}}\dot{\boldsymbol{\phi}}\right) \boldsymbol{v}_2 + \frac{1}{\sigma_3} \left(\boldsymbol{u}_3^{\mathrm{T}}\dot{\boldsymbol{\phi}}\right) \boldsymbol{v}_3 \tag{308}$$

It is seen that if $\boldsymbol{E}$ is close to a singularity, so that $\sigma_3$ tends to zero, then the amplification $1/\sigma_3$ of the component of $\dot{\boldsymbol{\phi}}$ along $\boldsymbol{u}_3$ will tend to infinity, and the resulting angular velocity component along $\boldsymbol{v}_3$ will tend to infinity.

This problem of the inverse relation at singularities can be solved with the damped least-squares solution. Then the inverse relation is modified to

$$\boldsymbol{\omega} = \frac{\sigma_1}{\sigma_1^2 + \lambda^2} \left(\boldsymbol{u}_1^{\mathrm{T}}\dot{\boldsymbol{\phi}}\right) \boldsymbol{v}_1 + \frac{\sigma_2}{\sigma_2^2 + \lambda^2} \left(\boldsymbol{u}_2^{\mathrm{T}}\dot{\boldsymbol{\phi}}\right) \boldsymbol{v}_2 + \frac{\sigma_3}{\sigma_3^2 + \lambda^2} \left(\boldsymbol{u}_3^{\mathrm{T}}\dot{\boldsymbol{\phi}}\right) \boldsymbol{v}_3 \tag{309}$$

If it is assumed that $\sigma_1 \geq \sigma_2 \gg \lambda$, then

$$\boldsymbol{\omega} = \frac{1}{\sigma_1} \left(\boldsymbol{u}_1^{\mathrm{T}}\dot{\boldsymbol{\phi}}\right) \boldsymbol{v}_1 + \frac{1}{\sigma_2} \left(\boldsymbol{u}_2^{\mathrm{T}}\dot{\boldsymbol{\phi}}\right) \boldsymbol{v}_2 + \frac{\sigma_3}{\sigma_3^2 + \lambda^2} \left(\boldsymbol{u}_3^{\mathrm{T}}\dot{\boldsymbol{\phi}}\right) \boldsymbol{v}_3 \tag{310}$$

This means that the solution is modified by damping the component resulting from the commanded motion in the direction associated with the singular value. This solution is an approximation which is accurate in directions 1 and 2.

### 8.12 Singularities of the kinematic differential equations for the ZYZ Euler angles

The kinematic differential equations for the $ZYZ$ Euler angles will have a singularity for $\sin\theta = 0$, which will cause the rotation axes of the $\psi$ rotation and the $\phi$ rotation to be aligned. This means that this Euler angle description will only give rotations about the $z$ and $y$ axes of frame 1. To analyze this in terms of an example the case $\theta = 0$ and $\psi = 0$. Then the kinematic differential equation $\boldsymbol{\omega} = \boldsymbol{F}\dot{\boldsymbol{\phi}}$ can be written

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{311}$$

where $\omega_x = 0$, $\omega_y = \dot{\theta}$ and $\omega_z = \dot{\psi} + \dot{\phi}$. There is no angular velocity about the $x$ axis. It is seen that this corresponds to the singularity for the roll-pitch-yaw angles.

### 8.13 Kinematic differential equations for quaternions

In this section the kinematic differential equations for the unit quaternion will be developed. Let $\boldsymbol{R} = \boldsymbol{R}_b^a$ be the rotation matrix from $a$ to $b$, and let $\boldsymbol{\omega}^a$ be the angular velocity in the coordinates of the $a$ frame. Time differentiation of the relation

$$\boldsymbol{R}\boldsymbol{a} = \boldsymbol{q} \circ \boldsymbol{a} \circ \bar{\boldsymbol{q}} \tag{312}$$

gives

$$\dot{\boldsymbol{R}}\boldsymbol{a} + \boldsymbol{R}\dot{\boldsymbol{a}} = \dot{\boldsymbol{q}} \circ \boldsymbol{a} \circ \bar{\boldsymbol{q}} + \boldsymbol{q} \circ \dot{\boldsymbol{a}} \circ \bar{\boldsymbol{q}} + \boldsymbol{q} \circ \boldsymbol{a} \circ \dot{\bar{\boldsymbol{q}}} \tag{313}$$

Here $\boldsymbol{R}\dot{\boldsymbol{a}} = \boldsymbol{q} \circ \dot{\boldsymbol{a}} \circ \bar{\boldsymbol{q}}$ and $\dot{\bar{\boldsymbol{q}}} = -\bar{\boldsymbol{q}} \circ \dot{\boldsymbol{q}} \circ \bar{\boldsymbol{q}}$. This gives

$$\begin{aligned} \dot{\boldsymbol{R}}\boldsymbol{a} =& \dot{\boldsymbol{q}} \circ \boldsymbol{a} \circ \bar{\boldsymbol{q}} - \boldsymbol{q} \circ \boldsymbol{a} \circ \bar{\boldsymbol{q}} \circ \dot{\boldsymbol{q}} \circ \bar{\boldsymbol{q}} & (314) \\ =& \dot{\boldsymbol{q}} \circ \bar{\boldsymbol{q}} \circ \boldsymbol{q} \circ \boldsymbol{a} \circ \bar{\boldsymbol{q}} - \boldsymbol{q} \circ \boldsymbol{a} \circ \bar{\boldsymbol{q}} \circ \dot{\boldsymbol{q}} \circ \bar{\boldsymbol{q}} & (315) \\ =& (\dot{\boldsymbol{q}} \circ \bar{\boldsymbol{q}}) \circ (\boldsymbol{R}\boldsymbol{a}) - (\boldsymbol{R}\boldsymbol{a}) \circ (\dot{\boldsymbol{q}} \circ \bar{\boldsymbol{q}}) & (316) \end{aligned}$$

This is the quaternion commutator of $\dot{\boldsymbol{q}} \circ \bar{\boldsymbol{q}}$ and $\boldsymbol{R}\boldsymbol{a}$. The quaternion commutator is given by (225), and in this case, this gives

$$\dot{\boldsymbol{R}}\boldsymbol{a} = 2(\dot{\boldsymbol{q}} \circ \bar{\boldsymbol{q}})^{\times}\boldsymbol{R}\boldsymbol{a} \tag{317}$$

Insertion of $\dot{\boldsymbol{R}} = (\boldsymbol{\omega}^a)^{\times}\boldsymbol{R}$ gives

$$(\boldsymbol{\omega}^a)^{\times}\boldsymbol{R}\boldsymbol{a} = 2(\dot{\boldsymbol{q}} \circ \bar{\boldsymbol{q}})^{\times}\boldsymbol{R}\boldsymbol{a} \tag{318}$$

This is valid for all values of $\boldsymbol{R}\boldsymbol{a}$, and it follows that

$$\boldsymbol{\omega}^a = 2\dot{\boldsymbol{q}} \circ \bar{\boldsymbol{q}} \tag{319}$$

which implies that $\dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{\omega}^a \circ \boldsymbol{q}$. Transformation to the $b$ frame where the anglar velcity is $\boldsymbol{\omega}^b$ is achieved with $\boldsymbol{\omega}^a = \boldsymbol{q}\boldsymbol{\omega}^b\bar{\boldsymbol{q}}$. This gives

$$\boldsymbol{\omega}^b = 2\boldsymbol{q} \circ \dot{\boldsymbol{q}} \tag{320}$$

which implies that $\dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{q} \circ \boldsymbol{\omega}^b$. To sum up, the kinematic differential equations are

$$\dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{\omega}^a \circ \boldsymbol{q} \tag{321}$$

$$\dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{q} \circ \boldsymbol{\omega}^b \tag{322}$$

This can be written out as

$$\dot{\boldsymbol{q}} = -\frac{1}{2}\boldsymbol{\epsilon}^{\mathrm{T}}\boldsymbol{\omega}^b + \frac{1}{2}(\eta\boldsymbol{I} - \boldsymbol{\epsilon}^{\times})\boldsymbol{\omega}^a \tag{323}$$

$$\dot{\boldsymbol{q}} = -\frac{1}{2}\boldsymbol{\epsilon}^{\mathrm{T}}\boldsymbol{\omega}^b + \frac{1}{2}(\eta\boldsymbol{I} + \boldsymbol{\epsilon}^{\times})\boldsymbol{\omega}^b \tag{324}$$

# 9 The Jacobian of a manipulator

## 9.1 Kinematics of an $n$-link manipulator

Consider a manipulator with $n$ links with joint variables $\boldsymbol{q} = [q_1, \ldots, q_n]^{\mathrm{T}}$, where the position and orientation of the end effector is given by the homogeneous transformation matrix

$$\boldsymbol{T}_e^0 = \begin{bmatrix} \boldsymbol{R}_e^0 & \boldsymbol{p}_{0e}^0 \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix} \tag{325}$$

where $\boldsymbol{R}_e^0$ is the rotation matrix from the base frame 0 to the end-effector frame $e$, and $\boldsymbol{p}_{0e}^0$ is the position of frame $e$ with respect to frame 0 in the coordinates of the 0 frame. The angular velocity $\boldsymbol{\omega}_{0e}^0$ of frame $e$ with respect to frame 0 in the coordinates of frame 0 is given by

$$\left(\boldsymbol{\omega}_{0e}^0\right)^{\times} = \dot{\boldsymbol{R}}_e^0(\boldsymbol{R}_e^0)^{\mathrm{T}} \tag{326}$$

where $\left(\boldsymbol{\omega}_{0e}^0\right)^{\times}$ is the skew-symmetric form $\boldsymbol{\omega}_{0e}^0$. The velocity $\boldsymbol{v}_e^0$ of the end effector in the coordinates of frame 0 is $\boldsymbol{v}_e^0 = \dot{\boldsymbol{p}}_{0e}^0$.

To simplify the notation we write the homogeneous transformation matrix as $\boldsymbol{T}_e = \boldsymbol{T}_e^0$, the rotation matrix as $\boldsymbol{R}_e = \boldsymbol{R}_e^0$, the position vector $\boldsymbol{p}_e = \boldsymbol{p}_{0e}^0$, the velocity vector $\boldsymbol{v}_e = \boldsymbol{v}_e^0$, and the angular velocity vector as $\boldsymbol{\omega}_e = \boldsymbol{\omega}_{0e}^0$. This gives

$$\boldsymbol{T}_e = \begin{bmatrix} \boldsymbol{R}_e & \boldsymbol{p}_e \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix} \tag{327}$$

and

$$\dot{\boldsymbol{T}}_e = \begin{bmatrix} \boldsymbol{\omega}_e^{\times}\boldsymbol{R}_e & \boldsymbol{v}_e \\ \boldsymbol{0}^{\mathrm{T}} & 0 \end{bmatrix} \tag{328}$$

### 9.2 The geometric Jacobian

The Jacobian $\boldsymbol{J}(\boldsymbol{q})$, which is a matrix that is a function of the joint variables $\boldsymbol{q}$, is used to describe the velocity and angular velocity of the end effector for a given joint velocity. This is useful in forward kinematics at the velocity level, and in the control of force and torque at the interaction between end effector and workpiece. Moreover, the Jacobian is important in certain techniques in inverse kinematics, where the inverse Jacobian is used. Of special importance are the singularities of the Jacobian $\boldsymbol{J}(\boldsymbol{q})$, which are joint positions where the Jacobian is not full rank, and therefore is not invertible. The singularities of the Jacobian are also referred to as the singularities of the manipulator, since the end effector of the manipulator will have a reduced number of freedom in a singularity, which imposes limitations on the dexterity of the manipulator and its ability to solve tasks at such configurations.

The basic form of the Jacobian is the geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$, which maps the joint velocities to the velocity and the angular velocity of the end effector. The is the the preferred form of the Jacobian. An alternative form of the Jacobian is the analytic Jacobian $\boldsymbol{J}_A(\boldsymbol{q})$ which is based on a vector description of the position and velocity of the end effector. This will typically mean the the orientation of the end effector is described in terms of roll-pitch-yaw Euler angles, whic leads to certain problems related to the Euler angle singularities which will be discussed in the following.

The geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ for a 6-link manipulator is a $6 \times 6$ matrix defined by

$$\begin{bmatrix} \dot{\boldsymbol{p}}_e \\ \boldsymbol{\omega}_e \end{bmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{329}$$

It is useful to write the Jacobian in terms of the velocity part and the angular velocity part as

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{J}_p(\boldsymbol{q}) \\ \boldsymbol{J}_o(\boldsymbol{q}) \end{bmatrix} \dot{\boldsymbol{q}} \tag{330}$$

which gives $\dot{\boldsymbol{p}}_e = \boldsymbol{J}_p(\boldsymbol{q})\dot{\boldsymbol{q}}$ and $\boldsymbol{\omega}_e = \boldsymbol{J}_o(\boldsymbol{q})\dot{\boldsymbol{q}}$.

### 9.3 Computation of the geometric Jacobian

In this section it is shown how the Jacobian can be efficiently computed with an algorithm that reduces the required hand calculation to a modest level.

The Denavit-Hartenberg convention is used, which means that all joint axes are along the $z$ axes of the Denavit-Hartenberg frames. The first step is to calculate all the link transformations

$$\boldsymbol{T}_i^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{331}$$

given the Denavit-Hartenberg parameters $\theta_i$, $d_i$, $\alpha_i$ and $a_i$ for $i = 1, \ldots, 6$.

The next step is to calculate

$$\boldsymbol{T}_2^0 = \boldsymbol{T}_1^0\boldsymbol{T}_2^1, \quad \boldsymbol{T}_3^0 = \boldsymbol{T}_2^0\boldsymbol{T}_3^2, \quad \boldsymbol{T}_4^0 = \boldsymbol{T}_3^0\boldsymbol{T}_4^3, \quad \boldsymbol{T}_5^0 = \boldsymbol{T}_4^0\boldsymbol{T}_5^4, \quad \boldsymbol{T}_6^0 = \boldsymbol{T}_5^0\boldsymbol{T}_6^5 \tag{332}$$

Then the joint axes $\boldsymbol{z}_i$ in frame 0 are found as the three first elements of column 3 of $\boldsymbol{T}_i^0$, and the position vector $\boldsymbol{p}_i$ of the origin of frame $i$ in the coordinates of frame 0 is found as the three first elements of column 4 of $\boldsymbol{T}_i^0$. Then the Jacobian can be computed from $\boldsymbol{z}_i$ and $\boldsymbol{p}_i$, $i = 1, \ldots, 6$.

**Example 1:**

If all the joints are rotational, and the Jacobian is the geometric Jacobian defined by

$$\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{333}$$

where $\boldsymbol{v}$ is the velocity and $\boldsymbol{\omega}$ is the angular velocity of the end effector in base coordinates, then the Jacobian is computed as

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{z}_0 \times (\boldsymbol{p}_6 - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p}_6 - \boldsymbol{p}_1) & \ldots & \boldsymbol{z}_5 \times (\boldsymbol{p}_6 - \boldsymbol{p}_5) \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \ldots & \boldsymbol{z}_5 \end{bmatrix} \tag{334}$$

In addition, the forward kinematics in terms of end-effector position and orientation is simply $\boldsymbol{T}_6^0$. It is interesting to note that this solution gives the forward kinematics in the form of an algorithm, and there is no complicated hand calculation involved. □

**Example 2:**

A SCARA manipulator has 4 joints. Therefore the configuration of the end effector will only have 4 degrees of freedom, which can be selected as $\boldsymbol{x} = [x, y, z, \psi]^{\mathrm{T}}$. The Denavit-Hartenberg parameters of the SCARA are given by

| Link | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ | |
|---|---|---|---|---|---|
| 1 | $a_1$ | 0 | $d_1$ | $\theta_1^*$ | $a_1 = 0.325$, $d_1 = 0.566$ |
| 2 | $a_2$ | 0 | 0 | $\theta_2^*$ | $a_2 = 0.225$ |
| 3 | 0 | 0 | $d_3^*$ | 0 | |
| 4 | 0 | 0 | $d_4$ | $\theta_4^*$ | $d_4 = -0.246$ |

where the joint variables are marked with an asterisk. The Jacobian defined by $\dot{\boldsymbol{x}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$ is then

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{z}_0 \times (\boldsymbol{p}_4 - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p}_4 - \boldsymbol{p}_1) & \boldsymbol{z}_2 & \boldsymbol{z}_3 \times (\boldsymbol{p}_4 - \boldsymbol{p}_3) \\ z_0 & z_1 & 0 & z_3 \end{bmatrix} \tag{335}$$

where $z_0$, $z_1$ and $z_3$ are the $z$ coordinates of the vectors $\boldsymbol{z}_0$, $\boldsymbol{z}_1$ and $\boldsymbol{z}_3$, respectively. □

## 9.4 The analytic Jacobian

Suppose that the position and orientation of the end effector is given by a six-dimensional vector

$$\boldsymbol{x}_e = \begin{bmatrix} \boldsymbol{p}_e \\ \boldsymbol{\phi}_e \end{bmatrix} \tag{336}$$

This can be done by letting the rotation matrix $\boldsymbol{R}_e$ be represented by the roll-pitch-yaw Euler angles $\boldsymbol{\phi}_e = [\phi, \theta\psi]^{\mathrm{T}}$. The position and orientation vector $\boldsymbol{x}_e$ is a function of the joint vector $\boldsymbol{q}$, which is written

$$\boldsymbol{x}_e = \boldsymbol{h}(\boldsymbol{q}) \tag{337}$$

The analytic Jacobian $\boldsymbol{J}_A$ is then defined by

$$\dot{\boldsymbol{x}}_e = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{338}$$

where the analytic Jacobian satisfies

$$\boldsymbol{J}_A(\boldsymbol{q}) = \left\{ \frac{\partial h_i}{\partial q_j} \right\} \tag{339}$$

The kinematic differential equation of the Euler angles is $\dot{\boldsymbol{\phi}}_e = \boldsymbol{E}\boldsymbol{\omega}_e$. This gives $\dot{\boldsymbol{\phi}}_e = \boldsymbol{E}\boldsymbol{\omega}_e = \boldsymbol{E}\boldsymbol{J}_o\dot{\boldsymbol{q}}$ where $\boldsymbol{J}_o$ is the orientation part of the geometric Jacobian. This means that the analytic Jacobian can be written

$$\boldsymbol{J}_A = \boldsymbol{G}\boldsymbol{J} \quad \text{where} \quad \boldsymbol{G} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{E} \end{bmatrix} \tag{340}$$

The singularities of $\boldsymbol{J}_A$ will therefore be the singularities of the geometric Jacobian $\boldsymbol{J}$ and, in addition, the singularities of $\boldsymbol{G}$. The singularities of $\boldsymbol{G}$ are the singularities of $\boldsymbol{E}$. This is the Euler angle singularity for the roll-pitch-yaw angles, which occurs for $\cos\theta = 0$. This is a serious drawback with the analytic Jacobian $\boldsymbol{J}_A$, as it has the additional singularity of the Euler angles in addition to the manipulator singularities, which are the singularities of the geometric Jacobian $\boldsymbol{J}$.

### 9.5 Virtual displacements

In statics and dynamics the concept of virtual displacements is an important tool.

### 9.6 Virtual displacements and the Jacobian

Consider a manipulator with $n$ links. The generalized coordinates of the manipulator is then a set of coordinates so that the position of every fixed point in the manipulator is known if the generalized coordinates are known. The joint variables $q_i$, $i = 1, \ldots, n$ will be a set of generalized coordinates. The position of the end effector is therefore a function $\boldsymbol{p}_e(q_1, \ldots, q_n)$ of the joint variables. The velocity $\dot{\boldsymbol{p}}_e$ and the angular velocity $\boldsymbol{\omega}_e$ of the end effector will satisfy

$$\dot{\boldsymbol{p}}_e = \boldsymbol{J}_p\dot{\boldsymbol{q}}, \quad \boldsymbol{\omega}_e = \boldsymbol{J}_o\dot{\boldsymbol{q}} \tag{341}$$

If a force vector $\boldsymbol{F}$ and a torque $\boldsymbol{M}$ is acting on the end effector, then the power of the force and torque is

$$P_e = \boldsymbol{F}^{\mathrm{T}}\dot{\boldsymbol{p}}_e + \boldsymbol{M}^{\mathrm{T}}\boldsymbol{\omega}_e \tag{342}$$

when the manipulator is moving with a velocity $\dot{\boldsymbol{p}}_e$ and angular velocity $\boldsymbol{\omega}_e$. The corresponding generalized joint forces are $\boldsymbol{\tau}$, so that $P_q = \boldsymbol{\tau}^{\mathrm{T}}\dot{\boldsymbol{q}}$. Then, from it follows from $P_e = P_q$ that

$$\boldsymbol{\tau}^{\mathrm{T}}\dot{\boldsymbol{q}} = \boldsymbol{F}^{\mathrm{T}}\dot{\boldsymbol{p}}_e + \boldsymbol{M}^{\mathrm{T}}\boldsymbol{\omega}_e = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{J}_p\dot{\boldsymbol{q}} + \boldsymbol{M}^{\mathrm{T}}\boldsymbol{J}_o\dot{\boldsymbol{q}}$$

which gives $\boldsymbol{\tau} = \boldsymbol{J}_p^{\mathrm{T}}\boldsymbol{F} + \boldsymbol{J}_o^{\mathrm{T}}\boldsymbol{M}$. This gives a very useful relation between the end-effector force and torque and the corresponding joint forces $\boldsymbol{\tau}$.

A shortcoming with the simple analysis above is that it is based on velocities. In the static case, the manipulator is not moving, and there is no velocity. Moreover, when the manipulator is moving with a certain velocity there will be acceleration terms that enter the force equations. Therefore the concept of virtual displacements is introduced. The virtual displacements of the joint variables are $\delta q_i$. The joint variables are the generalized coordinates, and can be changed freely. Therefore, the virtual displacements $\delta q_i$ can have any value. The virtual displacement of the end effector position must satisfy

$$\delta \boldsymbol{p}_e = \boldsymbol{J}_p \delta \boldsymbol{q} \tag{343}$$

The virtual displacement in rotation $\boldsymbol{\sigma}_e$ is slightly more complicated. The variation $\delta \boldsymbol{R}$ in the rotation matrix must satisfy the condition $\delta(\boldsymbol{R}\boldsymbol{R}^{\mathrm{T}}) = \delta \boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} + \boldsymbol{R}(\delta \boldsymbol{R})^{\mathrm{T}} = 0$. The virtual displacement is defined in its skew symmetric form as $\boldsymbol{\sigma}_e^{\times} = \delta \boldsymbol{R}\boldsymbol{R}^{\mathrm{T}}$. Then

$$\boldsymbol{\sigma}_e = \boldsymbol{J}_o \delta \boldsymbol{q} \tag{344}$$

The virtual work of a force $\boldsymbol{F}$ and a torque $\boldsymbol{M}$ acting on the end effector is defined as

$$\delta W_e = \boldsymbol{F}^{\mathrm{T}} \delta \boldsymbol{p} + \boldsymbol{M}^{\mathrm{T}} \boldsymbol{\sigma}_e = (\boldsymbol{F}^{\mathrm{T}} \boldsymbol{J}_p + \boldsymbol{M}^{\mathrm{T}} \boldsymbol{J}_o) \delta \boldsymbol{q}$$

The virtual work of the corresponding generalized joint force is $\delta W_q = \boldsymbol{\tau}^{\mathrm{T}} \delta \boldsymbol{q}$, and it follows from $\delta W_e = \delta W_q$ that

$$\boldsymbol{\tau} = \boldsymbol{J}_p^{\mathrm{T}} \boldsymbol{F} + \boldsymbol{J}_o^{\mathrm{T}} \boldsymbol{M} = \boldsymbol{J}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{M} \end{bmatrix} \tag{345}$$

In this case this was derived without introducing velocities, which means that the result can be used in a static analysis.

### 9.7 Statics

In statics the analysis is based on virtual displacements $\delta q_i$ of the joint variables $q_i$, where $\delta q_i$ is a arbitrary change in $q_i$ that is consistent with the linearized constraints of the system. The virtual work associated with a virtual displacement of the joint variables is then $\delta W_q = \boldsymbol{\tau}^{\mathrm{T}} \delta \boldsymbol{q}$.

To describe the virtual work of the end effector, the virtual displacement of the end effector must be described as

$$\boldsymbol{\xi} = \begin{bmatrix} \delta \boldsymbol{p} \\ \boldsymbol{\sigma} \end{bmatrix} \tag{346}$$

where $\delta \boldsymbol{p}$ is the virtual displacement in the position $\boldsymbol{p}$, while $\boldsymbol{\sigma}$ is the virtual displacement in rotation which satisfies $\delta \boldsymbol{R} = \boldsymbol{\sigma}^{\times} \boldsymbol{R}$. Note that $\dot{\boldsymbol{R}} = \boldsymbol{\omega}^{\times} \boldsymbol{R}$, which shows the relation between the angular velocity $\boldsymbol{\omega}$ and the virtual angular displacement $\boldsymbol{\sigma}$. Then if $\boldsymbol{\nu} = \boldsymbol{J}\dot{\boldsymbol{q}}$, it follows that

$$\boldsymbol{\xi} = \boldsymbol{J} \delta \boldsymbol{q} \tag{347}$$

Let the force and torque acting on the end effector be described with the vector

$$\gamma = \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{M} \end{bmatrix} \tag{348}$$

where $\boldsymbol{F}$ is the force acting on the end effector, and $\boldsymbol{M}$ is the moment so that the power that would result with a velocity $\boldsymbol{v}$ and an angular velocity $\boldsymbol{\omega}$ would be $P = \boldsymbol{F}^{\mathrm{T}} \boldsymbol{v} + \boldsymbol{M}^{\mathrm{T}} \boldsymbol{\omega}$. The virtual work of the end effector is then

$$\delta W_e = \boldsymbol{\gamma}^{\mathrm{T}} \boldsymbol{\xi} = \boldsymbol{F}^{\mathrm{T}} \delta \boldsymbol{p} + \boldsymbol{M}^{\mathrm{T}} \boldsymbol{\sigma} \tag{349}$$

The virtual work of the joint forces will be equal to the virtual work of the end-effector forces and torues, that is, $\delta W_q = \delta W_e$. This gives $\boldsymbol{\tau}^{\mathrm{T}}\delta\boldsymbol{q} = \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\xi}$. Insertion of (347) then gives

$$\boldsymbol{\tau} = \boldsymbol{J}^{\mathrm{T}}\boldsymbol{\gamma} \tag{350}$$

### 9.8 Geometric interpretation of nullspace and range space for a Jacobian

Consider a 6-link manipulator. The singular value decomposition of the $6 \times 6$ Jacobian $\boldsymbol{J}$ is given by

$$\boldsymbol{J} = \sum_{i=1}^{6} \sigma_i \boldsymbol{u}_i \boldsymbol{v}^{\mathrm{T}} \tag{351}$$

where $\boldsymbol{u}_i \in \mathbb{R}^6$, $i = 1, \ldots, 6$ are the orthogonal output vectors, and $\boldsymbol{v}_i \in \mathbb{R}^6$, $i = 1, \ldots, 6$ are the orthogonal input vectors. Then the velocity transformation $\boldsymbol{\nu} = \boldsymbol{J}\dot{\boldsymbol{q}}$ is written

$$\boldsymbol{\nu} = \boldsymbol{J}\dot{\boldsymbol{q}} = \sum_{i=1}^{r} \sigma_i (\boldsymbol{v}^{\mathrm{T}}\dot{\boldsymbol{q}})\boldsymbol{u}_i \tag{352}$$

where $r \leq 6$ is the rank of the Jacobian. It is seen that $\boldsymbol{\nu} \in \mathcal{R}(\boldsymbol{J}) = \mathrm{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r\}$, which is the range space of the Jacobian. Moreover, it is seen that if $\dot{\boldsymbol{q}} \in \mathcal{N}(\boldsymbol{J}) = \mathrm{span}\{\boldsymbol{v}_{r+1}, \ldots, \boldsymbol{v}_6\}$, which is the nullspace of the Jacobian, then $\boldsymbol{\nu} = 0$.

The force transformation is

$$\boldsymbol{\tau} = \boldsymbol{J}^{\mathrm{T}}\boldsymbol{\gamma} = \sum_{i=1}^{r} \sigma_i (\boldsymbol{u}^{\mathrm{T}}\boldsymbol{\gamma})\boldsymbol{v}_i \tag{353}$$

where $\boldsymbol{\tau}$ are the generalized forces of the joints, and $\boldsymbol{\gamma}$ are the corresponding generalized forces of the end effector, so that in the static case, $\boldsymbol{\tau}^{\mathrm{T}}\delta\boldsymbol{q} = \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\xi}$, where $\delta\boldsymbol{q}$ is the virtual displacement in $\boldsymbol{q}$, and $\boldsymbol{\xi} = \boldsymbol{J}\delta\boldsymbol{q}$ is the corresponding virtual displacement of the end effector. It is seen that $\boldsymbol{\tau} \in \mathcal{R}(\boldsymbol{J}^{\mathrm{T}}) = \mathrm{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$, which is the range space of the transposed Jacobian. Moreover, it is seen that if $\boldsymbol{\gamma} \in \mathcal{N}(\boldsymbol{J}^{\mathrm{T}}) = \mathrm{span}\{\boldsymbol{u}_{r+1}, \ldots, \boldsymbol{u}_6\}$, which is the nullspace of the transposed Jacobian, then $\boldsymbol{\tau} = 0$.

Consider the case where the Jacobian has rank $r = 6$ and is nonsingular. Then $\sigma_1 \geq \ldots \geq \sigma_6 > 0$, which means that all the singular values are greater than zero. In this case the range space is $\mathcal{R}(\boldsymbol{J}) = \mathrm{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_6\} = \mathbb{R}^6$. This means that $\boldsymbol{J}\dot{\boldsymbol{q}}$ can have any value, and all velocities $\boldsymbol{\nu}$ can be achieved. The nullspace is the empty space, and there is no joint velocity $\dot{\boldsymbol{q}}$ that gives zero output velocity $\boldsymbol{\nu}$. Concerning the force transformation, it is seen that there is no nonzero $\boldsymbol{\gamma}$ that will give a zero $\boldsymbol{\tau}$.

Next, consider the case where there is a singularity so that $\sigma_1 \geq \ldots \geq \sigma_5 > \sigma_6 = 0$, and the rank of $\boldsymbol{J}$ is $r = 5$. Then the range space is $\mathcal{R}(\boldsymbol{J}) = \mathrm{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_5\}$, and the nullspace is $\mathcal{N}(\boldsymbol{J}) = \mathrm{span}\{\boldsymbol{v}_6\}$. This means that the velocity $\boldsymbol{\nu} = \boldsymbol{J}\dot{\boldsymbol{q}}$ cannot have a component along $\boldsymbol{u}_6$, which means that the degrees of freedom of the end effector is reduced from 6 to 5 in the singularity. Moreover, joint velocities along $\boldsymbol{v}_6$ will be in the nullspace of $\boldsymbol{J}$, and will not give any velocity $\boldsymbol{\nu}$. This is seen by letting the joint velocities be given by $\dot{\boldsymbol{q}} = \alpha\boldsymbol{v}_6$ where $\alpha$ is a real scaling factor, which gives $\boldsymbol{\nu} = \boldsymbol{J}\alpha\boldsymbol{v}_6 = \sum_{i=1}^{5} \sigma_i (\boldsymbol{v}_i^{\mathrm{T}}\alpha\boldsymbol{v}_6)\boldsymbol{u}_i = \boldsymbol{0}$. In this case a end effector force $\boldsymbol{\gamma} = \beta\boldsymbol{u}_6$ along $\boldsymbol{u}_6$ will give a zero joint force vector since $\boldsymbol{\tau} = \boldsymbol{J}^{\mathrm{T}}\beta\boldsymbol{u}_6 = \sum_{i=1}^{5} \sigma_i (\boldsymbol{u}_i^{\mathrm{T}}\beta\boldsymbol{u}_6)\boldsymbol{v}_i = \boldsymbol{0}$.

This means that there are two effects of the singularity:

- In the velocity transformation there can be no end-effector velocity $\boldsymbol{\nu}$ along $\boldsymbol{u}_6$, and joint velocities $\dot{\boldsymbol{q}}$ along $\boldsymbol{v}_6$ give no end-effector velocity.
- In the force transformation, there can be no joint force $\boldsymbol{\tau}$ along $\boldsymbol{v}_6$, and an end-effector force $\boldsymbol{\gamma}$ along $\boldsymbol{u}_6$ gives zero joint forces.

# 10 Analytic inverse kinematics

## 10.1 Solving for the angle of a single joint

Consider a link of length $a$ that is rotated about the origin of the $xy$ plane by an angle $\theta$. The angle theta is zero when the link i aligned with the $x$ axis. The position of the tip point of the link is given by $x$ and $y$. The solution is then found with the Atan2 function as

$$\theta = \text{Atan2}(y, x) \tag{354}$$

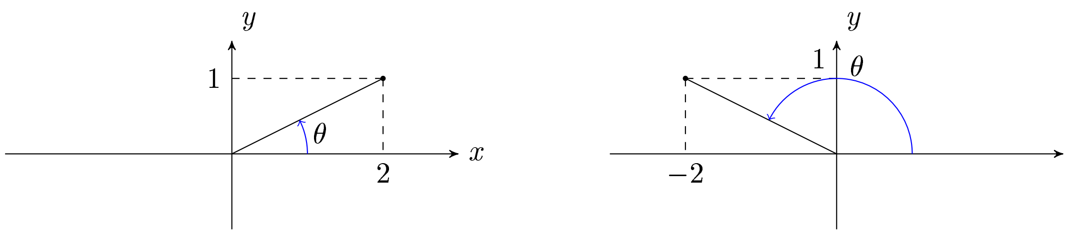



Figure 15: The Atan2 function when the first argument is positive will be in quadrant 1 or 2.

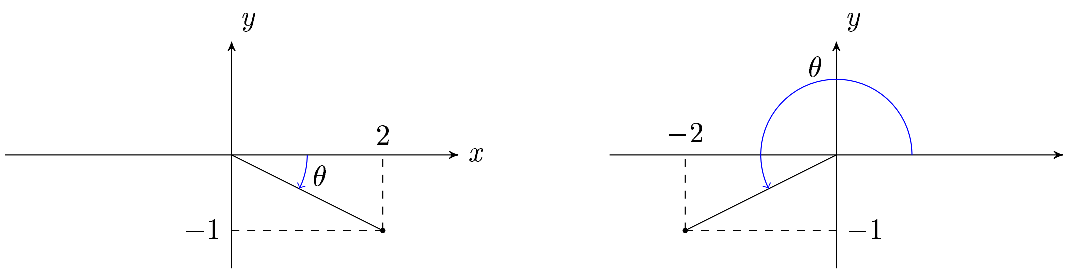



Figure 16: The Atan2 function when the first argument is negative will be in quadrant 3 or 4.

## 10.2 Solving for two joints in the plane using the law of cosines

Consider a planar manipulator with two joints with parallel joint axes. The inverse kinematic problem that is considered is to find the point angles $\theta_1$ and $\theta_2$ given the coordinates $x$ and $y$ of the the tip point. This is a standard problem in robotics, and is presented in, e.g., [16]. Let the link lengths be $a_1$ and $a_2$. Then the position of the tip will be

$$x = a_1 c_1 + a_2 c_{12} \tag{355}$$

$$y = a_1 s_1 + a_2 s_{12} \tag{356}$$

Squaring and summing gives

$$x^2 + y^2 = a_1^2(c_1^2 + s_1^2) + a_2^2(c_{12}^2 + s_{12}^2) + 2a_1a_2(c_1c_{12} + s_1s_{12}) \tag{357}$$

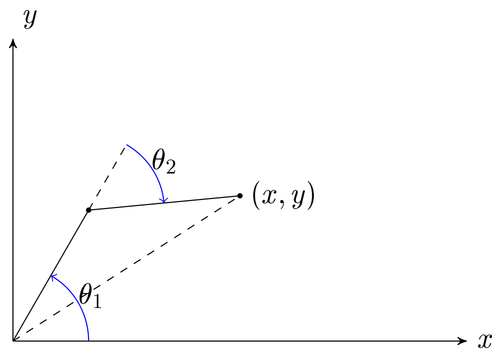



Figure 17: The geometry of the analytic inverse kinematics of the two links.

Using the trigonometric identity $c_1c_{12} + s_1s_{12} = c_2$ gives

$$x^2 + y^2 = a_1^2 + a_2^2 + 2a_1a_2c_2 \tag{358}$$

Then the cosine of the joint angle can be computed from

$$c_2 = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1a_2} \tag{359}$$

The angle $\theta_2$ can be found using the inverse cosine. It turns out to be better to use the Atan2 function. Then the sine of the angle is found from

$$s_2 = E\sqrt{1 - c_2^2} \tag{360}$$

and

$$\theta_2 = \text{Atan2}(s_2, c_2) \tag{361}$$

where $E = -1$ gives a solution with the elbow up, and $E = 1$ gives a solution with the elbow down.

To find the angle $\theta_1$ the tip position is reformulated using the geometric identities $c_{12} = c_1c_2 - s_1s_2$ and $s_{12} = s_1c_2 + c_2s_2$, which gives

$$x = (a_1 + a_2c_2)c_1 - a_2s_2s_1 \tag{362}$$

$$y = a_2s_2c_1 + (a_1 + a_2c_2)s_1 \tag{363}$$

This is a system of two linear equations in the two unknowns $c_1$ and $s_1$. Elimination of $c_1$ gives

$$(a_1 + a_2c_2)y - a_2s_2x = [a_2^2s_2^2 + (a_1 + a_2c_2)^2]s_1 \tag{364}$$

while elimination of $s_1$ gives

$$(a_1 + a_2c_2)x + a_2s_2y = [a_2^2s_2^2 + (a_1 + a_2c_2)^2]c_1 \tag{365}$$

It is straightforward to verify that $a_2^2s_2^2 + (a_1 + a_2c_2)^2 = a_1^2 + a_2^2 + 2a_1a_2c_2 = x^2 + y^2$. This gives the solution

$$s_1 = \frac{-a_2s_2x + (a_1 + a_2c_2)y}{x^2 + y^2} \tag{366}$$

$$c_1 = \frac{(a_1 + a_2c_2)x + a_2s_2y}{x^2 + y^2} \tag{367}$$

and the angle is found as

$$\theta_1 = \text{Atan2}(s_1, c_2) \tag{368}$$

An advantage with this solution is that when the logical variable $E$ is selected according to elbow up or down, the solutions for $\theta_1$ and $\theta_2$ will be consistent in the sense that both correspond to the same value for $E$.

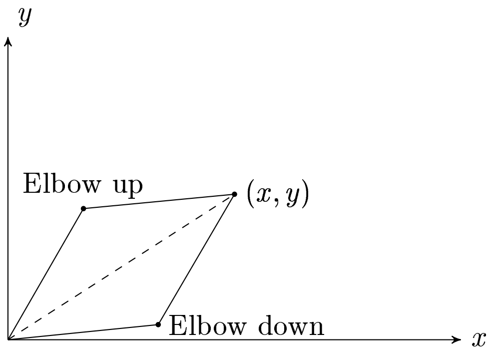



Figure 18: Two different solutions for the two links when the elbow is up and the elbow is down.

**Example**

Consider a two-link planar manipulator with link lengths $a_1 = a_2 = 0.5$. The tip position is given by $x = 0.5$ and $y = 0$. Then the solutions for the joint angles are given by

$$c_2 = \frac{x^2 + y^2 - a_2^2 - a_2^2}{2a_1a_2} = -0.5 \tag{369}$$

$$s_2 = E\sqrt{1 - c_2^2} = 0.8660E \tag{370}$$

When $E = 1$ the elbow is down and

$$s_1 = \frac{(a_1 + a_2c_2)y - a_2s_2x}{x^2 + y^2} = -0.8660 \tag{371}$$

$$c_1 = \frac{(a_1 + a_2c_2)x + a_2s_2y}{x^2 + y^2} = 0.5 \tag{372}$$

The angles are then

$$\theta_1 = \text{Atan2}(-0.8660, 0.5) = -60^\circ \tag{373}$$

$$\theta_2 = \text{Atan2}(0.8660, -0.5) = 120^\circ \tag{374}$$

When $E = -1$ the elbow is up and

$$s_1 = \frac{(a_1 + a_2c_2)y - a_2s_2x}{x^2 + y^2} = 0.8660 \tag{375}$$

$$c_1 = \frac{(a_1 + a_2c_2)x + a_2s_2y}{x^2 + y^2} = 0.5 \tag{376}$$

and the angles are

$$\theta_1 = \text{Atan2}(0.8660, 0.5) = 60^\circ \tag{377}$$

$$\theta_2 = \text{Atan2}(-0.8660, -0.5) = -120^\circ \tag{378}$$

Note that the solutions for $\theta_1$ and $\theta_2$ are consistent, as the selected value for $\theta_2$ is used in the computation of $\theta_1$.

```
% Calculation of the joint angles for a two-link planar manipultor
a1 = 0.5; a2 = 0.5;
x = 0.5; y = 0.;
E = -1 % Elbow up

c2 = (x^2 + y^2 - a1^2 - a2^2)/(2*a1*a2)
s2 = E*sqrt(1-c2^2)
q2 = atan2d(s2,c2)

s1 = (- a2*s2*x + (a1+a2*c2)*y )/(x^2+y^2)
c1 = ((a1+a2*c2)*x + a2*s2*y)/(x^2+y^2)
q1 = atan2d(s1,c1)
```

□

### 10.3 Inverse kinematics of an anthropomorphic arm with a spherical wrist

In this section the singularities of an anthropomorphic arm with a spherical wrist is analyzed. Suppose that the position and orientation of frame 6 is given by

$$\boldsymbol{T}_6^0 = \begin{bmatrix} \boldsymbol{n}_6 & \boldsymbol{s}_6 & \boldsymbol{a}_6 & \boldsymbol{p}_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{379}$$

In this case the three wrist axes intersect at the wrist point. The inverse kinematics is then possible to solve in two steps. First the position of the wrist point is calculated. This is done by observing that the origin of frame 5 is at the wrist point. This means that the position of the origin of frame 6 has position $d_6\boldsymbol{a}_6$ relative to the wrist point, where $d_6$ is the Denavit Hartenberg parameter giving the translation along the $z$ axis from frame 5 to frame 6, and $\boldsymbol{a}_6$ is the vector along the $z$ axis of frame 6. This means that the position of the wrist point is found from $\boldsymbol{p}_w = \boldsymbol{p}_6 - d_6\boldsymbol{a}_6$.

When $\boldsymbol{p}_w$ has been calculated, the joint angles $\theta_1$, $\theta_2$ and $\theta_3$ can be found. Using these angles, it is possible to calculate $\boldsymbol{R}_3^0$, which is used to find $\boldsymbol{R}_6^3 = (\boldsymbol{R}_3^0)^{\mathrm{T}}\boldsymbol{R}_6^0$. Then, the wrist angles $\theta_4$, $\theta_5$ and $\theta_6$ will be a set of Euler angles corresponding to $\boldsymbol{R}_6^3$, and these angles are found using the usual techniques for the calculation of Euler angles.

### 10.4 Inverse kinematics of the ABB IRb 2000

The analytic inverse kinematics of the ABB IRb 2000 is presented in this section. The Denavit-Hartenberg parameters of the manipulator are given by

| Link | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | $-\pi/2$ | 0.750 | $\theta_1^*$ |
| 2 | 0.710 | 0 | 0 | $\theta_2^*$ |
| 3 | 0.125 | $-\pi/2$ | 0 | $\theta_3^*$ |
| 4 | 0 | $\pi/2$ | 0.850 | $\theta_4^*$ |
| 5 | 0 | $-\pi/2$ | 0 | $\theta_5^*$ |
| 6 | 0 | 0 | 0.100 | $\theta_6^*$ |

The home position of the manipulator is set to $\boldsymbol{q}_h = [0, -\pi/2, 0, 0, 0, 0]^{\mathrm{T}}$.

The manipulator has spherical wrist. The standard method for solving the analytic inverse kinematics is therefore based on finding the wrist point, and use this to calculate the joint angles $\theta_1$, $\theta_2$ and $\theta_3$. From these angles the transformation matrix $\boldsymbol{T}_0^3$ can be calculated, and the wrist angles $\theta_4$, $\theta_5$ and $\theta_5$, which are three Euler angles, can be calculated from $\boldsymbol{T}_6^3 = (\boldsymbol{T}_0^3)^{\mathrm{T}}\boldsymbol{T}_6^0$.

Let the position and orientation of the end effector frame $e$ be given by

$$\boldsymbol{T}_e^0 = \begin{bmatrix} \boldsymbol{n}_e & \boldsymbol{s}_e & \boldsymbol{a}_e & \boldsymbol{p}_e \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{380}$$

Let the origin of Denavit-Hartenberg frame $i$ be denoted $\boldsymbol{p}_i$, and let the unit coordinate vectors along the $x$, $y$ and $z$ axes of frame $i$ be denoted $\boldsymbol{x}_i$, $\boldsymbol{y}_i$ and $\boldsymbol{z}_i$, respectively, where the vectors are given in the coordinates of frame 0. It is assumed that the $e$ frame is frame 6 in the Denavit-Hartenberg description.

The manipulator has three logical variables used to select different configurations. The End effector is in the front when $F = 1$ and in the back when $F = -1$. The elbow is up when $\boldsymbol{E}_1$ and down when $E = -1$. The wrist is flipped when $W = 1$ and not flipped when $W = -1$.

The wrist point will have the position

$$\boldsymbol{p}_w = \boldsymbol{p}_e - d_6\boldsymbol{a}_e \tag{381}$$

with coordinates $\boldsymbol{p}_w = [x_w, y_w, z_w]^{\mathrm{T}}$. Then the angle of joint 1 is found from

$$\theta_1 = \mathrm{Atan2}(Fy_w, Fx_w) \tag{382}$$

Joint angles $\theta_2$ and $\theta_3$ are calculated as for a planar two-link manipulator. The length

$$L_3 = \sqrt{a_3^2 + d_4^2} \tag{383}$$

and the angle $\psi_3$ are defined to handle the offset $a_3$, so that $\theta_3 = \psi_3 + \delta_3$, where the offset angle

$$\delta_3 = \mathrm{Atan2}(d_4, a_3) \tag{384}$$

is the the offset angle, so that

$$\theta_3 = \psi_3 - \delta_3 \tag{385}$$

The triangle of the angles $\theta_2$ and $\psi_3$ has the lengths

$$p_h = F\sqrt{x_w^2 + y_w^2} \tag{386}$$

$$p_v = -(z_w - d_1) \tag{387}$$

Then

$$c_3 = \frac{p_h^2 + p_v^2 - a_2^2 - L_3^2}{2a_2L_3} \tag{388}$$

$$s_3 = E\sqrt{1 - c_3^2} \tag{389}$$

$$\psi_3 = \text{Atan2}(s_3, c_3) \tag{390}$$

and

$$c_2 = \frac{p_h(a_2 + L_3c_3) + p_vL_3s_3}{p_h^2 + p_v^2} \tag{391}$$

$$s_2 = \frac{p_v(a_2 + L_3c_3) - p_hL_3s_3}{p_h^2 + p_v^2} \tag{392}$$

$$\theta_2 = \text{Atan2}(s_2, c_2) \tag{393}$$

The computed values for $\theta_1$, $\theta_2$ and $\theta_3$ can then be used to compute the rotation matrix $\boldsymbol{R}_0^3$. Then the wrist angles can be computed from $\boldsymbol{R}_6^3 = (\boldsymbol{R}_0^3)^{\mathrm{T}}\boldsymbol{R}_6^0$.

The joint angles will be a set of ZYZ Euler angles $\psi$, $\theta$ and $\phi$, where the first angle $\psi$ is about the the $z_3$ axis, the second angle $\theta$ is about $-\boldsymbol{z}_4$, and the third angle $\phi$ is about $z_5$. This means that $\theta_4 = \psi$, $\theta_5 = -\theta$ and $\theta_6 = \phi$. The solution is therefore

$$\theta_4 = \text{Atan2}(Wr_{23}, Wr_{13}) \tag{394}$$

$$\theta_5 = -\text{Atan2}(W\sqrt{r_{13}^2 + r_{23}^2}, r_{33}) \tag{395}$$

$$\theta_6 = \text{Atan2}(Wr_{23}, -Wr_{31}) \tag{396}$$

where $\boldsymbol{R}_6^3 = \{r_{ij}\}$ where $W = -1$ when the wrist is not flipped, and $W = 1$ when the wrist is flipped.

### 10.5 Singularities in an industrial manipulator

In this section the singularities of the ABB IRb 2000 are analyzed following the presentation in [2]. This manipulator has three singularities.

- The elbow singularity occurs when the elbow in joint 3 is stretched out. Joints 2 and 3 will in the nonsingular case move the the wrist point in two degrees of freedom in the plane that is perpendicular to joint axes 2 and 3. In the singularity when the elbow is stretched out, the wrist point can only be moved in one degree of freedom in this plane. The condition for the elbow singularity is $a_3 \cos\theta_3 + d_4 \sin\theta_3 = 0$. This singularity occurs outside of the joint limit for $\theta_3$, so it has no practical interest for this manipulator.
- The shoulder singularity occurs when the wrist point is at the rotation axis of joint 1. Then there can be no velocity of the wrist point in the direction of joint axes 2 and 3. The condition for this singularity is $a_2 \sin\theta_2 + a_3 \cos(\theta_2 + \theta_3) + d_4 \sin(\theta_2 + \theta_3) = 0$.
- The wrist singularity has the condition $q_5 = 0$. In this case the joint axes of joints 4 and 6 will be aligned, and there can be no rotation from the wrist about the axis that is orthogonal to joint axes 4,5 and 6.

This manipulator has a spherical wrist with three intersecting wrist axes. The inverse kinematics for this type of manipulator can be solved by first finding the wrist point from the end effector position and orientation. Then the joint angles of joints 1, 2 and 3 are found given the position of the wrist. Finally, the wrist angles are found given the rotation matrix $\boldsymbol{R}_3^6$ from frame 3 to frame 6, which is found from the the angles of joints 1, 2 and 3 and the orientation of the end effector. Note that this requires that the wrist position can be positioned in 3 degrees of freedom by joints 1, 2 and 3, and the rotation $\boldsymbol{R}_3^6$ can be achieved with the wrist angles 4, 5 and 6.

In the elbow and shoulder singularity, the singularity can be identified from the fact that the wrist motion will not have 3 degrees of freedom in these singularities. Then the end effector will have at most 5 degrees of freedom, which means that it is at a singularity. For the shoulder singularity, the singularity is identified because the wrist angles will only provide two degrees of freedom for the rotation $\boldsymbol{R}_3^6$.

It is noted that the manipulator can have a combination of singularities. If the manipulator has the wrist point on the rotation axis of joint axis 1, and $q_5 = 0$, then it can be said to be at a double singularity. The position of the wrist point will then have two degrees of freedom, and the wrist joints will only provide 2 rotational degrees of freedom, so the end effector will have at most 4 degrees of freedom. A triple singularity will occur if in addition the elbow is stretched out. Then the wrist point will have only one degree of freedom, which is a velocity in the horizontal direction perpendicular to joint axes 2 and 3. The wrist will only have two degrees of freedom, and it follows that the end effector will have 3 degrees of freedom.

### 10.6 Inverse kinematics of the KUKA Agilus

The Denavit-Hartenberg parameters of the KUKA Agilus manipulator are

| Link | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_{i0}$ |
|---|---|---|---|---|
| 1 | 0.025 | $-\pi/2$ | 0.400 | 0 |
| 2 | 0.455 | 0 | 0 | 0 |
| 3 | 0.035 | $-\pi/2$ | 0 | $-\pi/2$ |
| 4 | 0 | $\pi/2$ | 0.420 | 0 |
| 5 | 0 | $-\pi/2$ | 0 | 0 |
| 6 | 0 | 0 | 0.08 | 0 |

The manipulator has 6 rotary joints. Here $\theta_{i0}$ is the offset in joint angle $i$. This means that the Denavit-Hartenberg joint angle $\theta_i$ is related to the physical joint angle $q_i$ by $\theta_i = q_i + \theta_{0i}$. The home position of the manipulator is set to $\boldsymbol{q}_h = [0, -\pi/2, \pi/2, 0, 0, 0]^{\mathrm{T}}$. The manipulator is shown in the home position in Figure 19.

The KUKA Agilus robot has spherical wrist. Then the standard method for solving the analytic inverse kinematics is based on finding the wrist point, and use this to calculate the joint angles $\theta_1$, $\theta_2$ and $\theta_3$. From these angles the transformation matrix $\boldsymbol{T}_0^3$ can be calculated, and the wrist angles $\theta_4$, $\theta_5$ and $\theta_5$, which are three Euler angles, can be calculated from $\boldsymbol{T}_6^3 = (\boldsymbol{T}_0^3)^{\mathrm{T}}\boldsymbol{T}_6^0$.

Let the position and orientation of the end effector frame $e$ be given by

$$\boldsymbol{T}_e^0 = \begin{bmatrix} \boldsymbol{n}_e & \boldsymbol{s}_e & \boldsymbol{a}_e & \boldsymbol{p}_e \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{397}$$

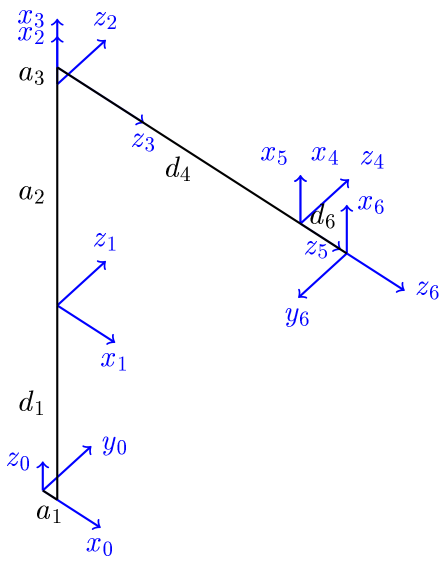



Figure 19: The Denavit-Hartenberg frames of the KUKA Agilus robot in the home position $\boldsymbol{q}_h = [0, -\pi/2, \pi/2, 0, 0, 0]^{\mathrm{T}}$.

Let the origin of Denavit-Hartenberg frame $i$ be denoted $\boldsymbol{p}_i$, and let the unit coordinate vectors along the $x$, $y$ and $z$ axes of frame $i$ be denoted $\boldsymbol{x}_i$, $\boldsymbol{y}_i$ and $\boldsymbol{z}_i$, respectively, where the vectors are given in the coordinates of frame 0. It is assumed that the $e$ frame is frame 6 in the Denavit-Hartenberg description.

The manipulator has three logical variables used to select different configurations. The end effector is in the front when $F = 1$ and in the back when $F = -1$. The elbow is up when $E_1$ and down when $E = -1$. The wrist is flipped when $W = 1$ and not flipped when $W = -1$.

The wrist point will have the position

$$\boldsymbol{p}_w = \boldsymbol{p}_e - d_6 \boldsymbol{a}_e \tag{398}$$

with coordinates $\boldsymbol{p}_w = [x_w, y_w, z_w]^{\mathrm{T}}$. Then

$$\theta_1 = \mathrm{Atan2}(F y_w, F x_w) \tag{399}$$

Joint angles $\theta_2$ and $\theta_3$ are calculated as for a planar two-link manipulator. The length

$$L_3 = \sqrt{a_3^2 + d_4^2} \tag{400}$$

and the angle $\psi_3$ are defined to handle the offset $a_3$, where $\theta_3 = \psi_3 \pm \delta_3$, where the offset angle

$$\delta_3 = \mathrm{Atan2}(a_3, d_4) \tag{401}$$

is the the offset angle, so that

$$\theta_3 = \psi_3 + \delta_3 \tag{402}$$

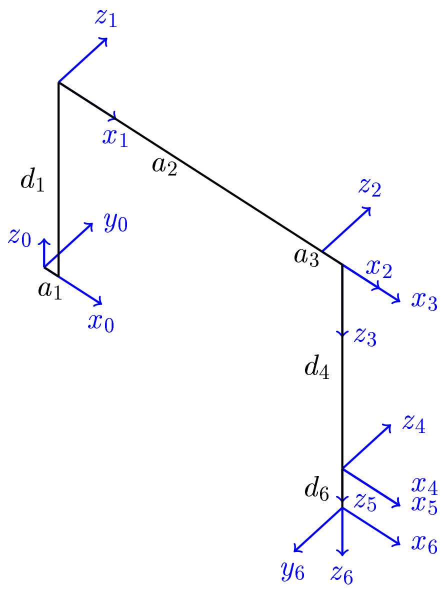



Figure 20: The Denavit-Hartenberg frames of the KUKA Agilus robot in the zero position $\boldsymbol{q}_h = [0, 0, \pi/2, 0, 0, 0]^{\mathrm{T}}$ where all the joint angles are zero. Note that joint angle 3 has an offset of $\pi/2$.

The triangle of the angles $\theta_2$ and $\psi_3$ has horizontal length $p_h = F\sqrt{x_w^2 + y_w^2} - Fa_1$ and $p_v = -(z_w - d_1)$. Then

$$c_3 = \frac{p_h^2 + p_v^2 - a_2^2 - L_3^2}{2a_2 L_3} \tag{403}$$

$$s_3 = E\sqrt{1 - c_3^2} \tag{404}$$

$$\psi_3 = \text{Atan2}(s_3, c_3) \tag{405}$$

and

$$c_2 = \frac{p_h(a_2 + L_3 c_3) + p_v L_3 s_3}{p_h^2 + p_v^2} \tag{406}$$

$$s_2 = \frac{p_v(a_2 + L_3 c_3) - p_h L_3 s_3}{p_h^2 + p_v^2} \tag{407}$$

$$\theta_2 = \text{Atan2}(s_2, c_2) \tag{408}$$

The computed values for $\theta_1$, $\theta_2$ and $\theta_3$ can then be used to compute the rotation matrix $\boldsymbol{R}_0^3$. Then the wrist angles can be computed from $\boldsymbol{R}_6^3 = (\boldsymbol{R}_0^3)^{\mathrm{T}} \boldsymbol{R}_6^0$. The joint angles will be a set of ZYZ Euler angles $\psi$, $\theta$ and $\phi$, where the first angle $\psi$ is about the the $z_3$ axis, the second angle $\theta$ is about $-\boldsymbol{z}_4$, and the third angle $\phi$ is about $z_5$. This means that $\theta_4 = \psi$, $\theta_5 = -\theta$ and $\theta_6 = \phi$. The solution is therefore

$$\theta_4 = \text{Atan2}(Wr_{23}, Wr_{13}) \tag{409}$$

$$\theta_5 = -\text{Atan2}(W\sqrt{r_{13}^2 + r_{23}^2}, r_{33}) \tag{410}$$

$$\theta_6 = \text{Atan2}(Wr_{23}, -Wr_{31}) \tag{411}$$

where $\boldsymbol{R}_6^3 = \{r_{ij}\}$ where $W = -1$ when the wrist is not flipped, and $W = 1$ when the wrist is flipped.

**MATLAB Example**

```
% Script for testing the inverse kinematics of the KUKA Agilus
clear;
SetDH_Agilus; % Set Denavit Hartenvberg parameters
% Input hoint angles
qh = [pi   -pi/2 pi/2  0  0  0]'; % Home position
q = qh - [0 0 0 0 0.2 0]' % Offset from home position

[J, T] = ForwardKinAgilus(dh, q) % Forward kinematics

qa = AInverseKinAg(dh,T) % Analytic inverse kinemtics
q_inv_solution = qa'
q_actual = q'
[Ja, Ta] = ForwardKinAgilus(dh, qa); % Test if solution is correct
Ta-T % Zero Matrix expected

%%%%%%%%%%%%%%%%%%%

function q = AInverseKinAg(dh, Te)
% Analytic inverse kimenatics for the KUKA Agilus

ne = Te(1:3,1); se = Te(1:3,2); ae = Te(1:3,3);
pe = Te(1:3,4);

d1 = dh(1,3); d4 = dh(4,3); d6 = dh(6,3);
a1 = dh(1,1); a2 = dh(2,1); a3 = dh(3,1); L3 = sqrt(a3^2 + d4^2);

pw = pe - d6*ae;

FRONT = 1;
q1 = atan2(FRONT*pw(2),FRONT*pw(1));

ELBOW = 1;
pty = - (pw(3) - d1);
ptx = FRONT*(sqrt(pw(1)^2+pw(2)^2)-FRONT*a1);

c3 = (ptx^2 + pty^2 - a2^2 - L3^2) / (2*a2*L3);
s3 = ELBOW*sqrt(1-c3^2);
psi3 = atan2(s3,c3);
offset3 = atan2(a3,d4);
q3 = psi3 + offset3;

c2 = (ptx*(a2 + L3*c3) + pty*L3*s3) / (ptx^2 + pty^2);
```

```
s2 = (pty*(a2 + L3*c3) - ptx*L3*s3) / (ptx^2 + pty^2);
q2 = atan2(s2,c2);

q = [q1;q2;q3;0;0;0];
% dh = [a alpha d theta]
dhf = dh;
% Insert joint variable in DH vectors
for i = 1:3
    dhf(i,4) = dhf(i,4) + q(i);
end
% Homogeneous matrices for links
T01 =  LinkDH2T (dhf(1,:));
T12 =  LinkDH2T (dhf(2,:));
T23 =  LinkDH2T (dhf(3,:));
T02 = T01 * T12;
T03 = T02 * T23;
R03 = T03(1:3,1:3); Re = Te(1:3,1:3);
R = R03'*Re

WRIST = 1;
q4 = atan2(WRIST*R(2,3),WRIST*R(1,3));
q5 = -atan2(WRIST*sqrt(R(1,3)^2+R(2,3)^2),R(3,3));
q6 = atan2(WRIST*R(3,2),-WRIST*R(3,1));

q(4:6) = [q4; q5; q6];

%%%%%%%%%%%%

function [J, Te] = ForwardKinAgilus(dh, q)
% dh = [a alpha d theta]

dhf = dh;
% Insert joint variable in DH vectors
for i = 1:6
    dhf(i,4) = dhf(i,4) + q(i);
end

% Homogeneous matrices for links
T01 =  LinkDH2T (dhf(1,:));
T12 =  LinkDH2T (dhf(2,:));
T23 =  LinkDH2T (dhf(3,:));
T34 =  LinkDH2T (dhf(4,:));
T45 =  LinkDH2T (dhf(5,:));
T56 =  LinkDH2T (dhf(6,:));

% Homogeneous matrices for intermediate frames
```

```
T02 = T01 * T12;
T03 = T02 * T23;
T04 = T03 * T34;
T05 = T04 * T45;

% Homogeneous matrix for manipulator
Te = T05 * T56;

% Rotation axes
z0 = [0 0 1]';
z1 = T01(1:3,3);
z2 = T02(1:3,3);
z3 = T03(1:3,3);
z4 = T04(1:3,3);
z5 = T05(1:3,3);

% Position vectors
p0 = [0 0 0]';
p1 = T01(1:3,4);
p2 = T02(1:3,4);
p3 = T03(1:3,4);
p4 = T04(1:3,4);
p5 = T05(1:3,4);

pe = Te(1:3,4);

% Geometric Jacobian for rotational joints
  J = [ cross(z0,(pe-p0)) cross(z1,(pe-p1)) cross(z2,(pe-p2)) ...
      cross(z3,(pe-p3)) cross(z4,(pe-p4)) cross(z5,(pe-p5)) ;
      z0 z1 z2 z3 z4 z5 ];

%%%%%%%%%%%%%%%%

function T = LinkDH2T(dh)
% dh = [a alpha d theta]

xv = [1;0;0]; zv=[0;0;1];
T =  AngleAxis2T(zv,dh(4)) * Trans2T(dh(3)*zv) ...
     * AngleAxis2T(xv,dh(2)) * Trans2T(dh(1)*xv);

%%%%%%%%%%%%%%%
```

### 10.7 Singularities of the KUKA Agilus industrial robot

In this section the singularities of the KUKA Agilus KR 6 R900 sixx are analyzed. This manipulator has three singularities.

- The elbow singularity occurs when the elbow in joint 3 is stretched out. Joints 2 and 3 will in the nonsingular case move the wrist point in two degrees of freedom in the plane that is perpendicular to joint axes 2 and 3. In the singularity when the elbow is stretched out, the wrist point can only be moved in one degree of freedom in this plane. The condition for the elbow singularity is $a_3 \cos q_3 + d_4 \sin q_3 = 0$, which gives $q_3 = \arctan(a_3/d_4) = 0.0831$ rad.
- The shoulder singularity occurs when the wrist point is at the rotation axis of joint 1. Then there can be no velocity of the wrist point in the direction of joint axes 2 and 3. The condition for this singularity is $a_1 + a_2 \sin q_2 + a_3 \cos(q_2 + q_3) + d_4 \sin(q_2 + q_3) = 0$. If $q_3 = \pi/2$, then the shoulder singularity occurs for $q_2 = -2.3182$. If the robot is in the elbow singularity with $q_3 = 0.0831$ rad, then it will also be in the shoulder singularity if $q_2 = -1.5995$.
- The wrist singularity has the condition $q_5 = 0$. In this case the joint axes of joints 4 and 6 will be aligned, and there can be no rotation from the wrist about the axis that is orthogonal to joint axes 4,5 and 6.

This manipulator has a spherical wrist with three intersecting wrist axes. The inverse kinematics for this type of manipulator can be solved by first finding the wrist point from the end effector position and orientation. Then the joint angles of joints 1, 2 and 3 are found given the position of the wrist. Finally, the wrist angles are found given the rotation matrix $\boldsymbol{R}_3^6$ from frame 3 to frame 6, which is found from the the angles of joints 1, 2 and 3 and the orientation of the end effector. Note that this requires that the wrist position can be positioned in 3 degrees of freedom by joints 1, 2 and 3, and the rotation $\boldsymbol{R}_3^6$ can be achieved with the wrist angles 4, 5 and 6.

In the elbow and shoulder singularity, the singularity can be identified from the fact that the wrist motion will not have 3 degrees of freedom in these singularities. The the end effector will have at most 5 degrees of freedom, which means that it is at a singularity. For the shoulder singularity, the singularity is identified because the wrist angles will only provide two degrees of freedom for the rotation $\boldsymbol{R}_3^6$.

It is noted that the manipulator can have a combination of singularities. If the manipulator has the wrist point on the rotation axis of joint axis 1, and $q_5 = 0$, then it can be said to be at a double singularity. The position of the wrist point will then have two degrees of freedom, and the wrist joints will only provide 2 rotational degrees of freedom, so the end effector will have at most 4 degrees of freedom.

### 10.8 Inverse kinematics of the UR5 industrial robot

In this section the analytic inverse kinematics for the Universal UR5 are presented. The derivation is based on the geometric insight gained from the approach used in [8] where the solution is found using geometric algebra.

The Denavit-Hartenberg parameters of the Universal UR5 robot are

| Link | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ | |
|---|---|---|---|---|---|
| 1 | 0 | $\frac{\pi}{2}$ | $d_1$ | $\theta_1^*$ | $d_1 = 0.0892$ |
| 2 | $a_2$ | 0 | 0 | $\theta_2^*$ | $a_2 = -0.425$ |
| 3 | $a_3$ | 0 | 0 | $\theta_3^*$ | $a_3 = -0.39243$ |
| 4 | 0 | $\frac{\pi}{2}$ | $d_4$ | $\theta_4^*$ | $d_4 = 0.109$ |
| 5 | 0 | $-\frac{\pi}{2}$ | $d_5$ | $\theta_5^*$ | $d_5 = 0.093$ |
| 6 | 0 | 0 | $d_6$ | $\theta_6^*$ | $d_6 = 0.082$ |

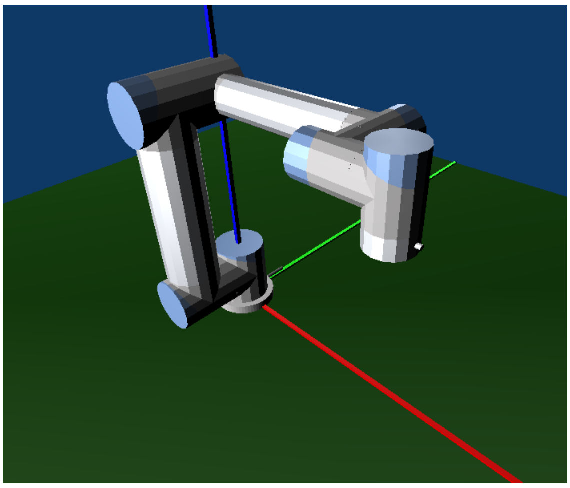


Figure 21: Universal Robot UR5 in home configuration

The home position of the manipulator as shown i Figure 21 is given by the joint angles

$$\boldsymbol{q}_{\text{home}} = \begin{pmatrix} 0 & -\frac{\pi}{2} & -\frac{\pi}{2} & -\frac{\pi}{2} & \frac{\pi}{2} & 0 \end{pmatrix}^{\mathrm{T}}.$$

The Denavit-Hartenberg frames of the UR5 robot in the home position $\boldsymbol{q}_{\text{home}}$ are shown in the Figure 22.

Let the position and orientation of the end effector frame $e$ be given by

$$\boldsymbol{T}_e^0 = \begin{bmatrix} \boldsymbol{n}_e & \boldsymbol{s}_e & \boldsymbol{a}_e & \boldsymbol{p}_e \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{412}$$

Let the origin of Denavit-Hartenberg frame $i$ be denoted $\boldsymbol{p}_i$, and let the unit coordinate vectors along the $x$, $y$ and $z$ axes of frame $i$ be denoted $\boldsymbol{x}_i$, $\boldsymbol{y}_i$ and $\boldsymbol{z}_i$, respectively, where the vectors are given in the coordinates of frame 0. It is assumed that the $e$ frame is frame 6 in the Denavit-Hartenberg description.

The inverse kinematic problem is then to find the joint angles $\boldsymbol{q}$ given $\boldsymbol{T}_0^e$. The Universal Robots UR5 manipulator does not have a spherical wrist, so there is no wrist point that can decouple the inverse kinematic solution into a position part for joints 1, 2 and 3 and and orientation part for joints 4, 5 an 6. The solution can still be found because joints 2, 3 and 4 have parallel axes of rotation.

The home position is characterized by as a configuration with the shoulder to the left, the elbow up, and the wrist pointing away from the base frame. The inverse kinematics will first

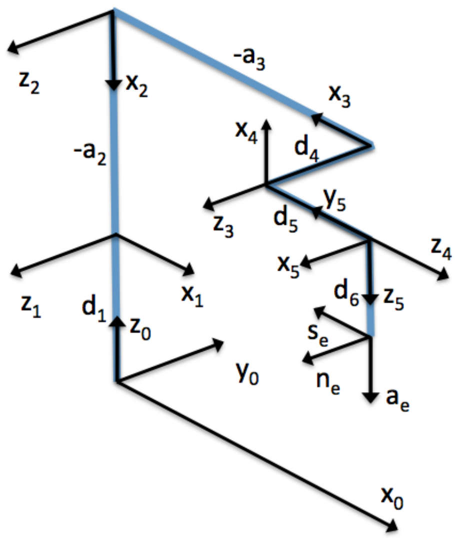



Figure 22: Universal Robot UR5 with Denavit-Hartenberg frames in home configuration

be found for this type of configuration. Then, the solution will be extended to include shoulder right, elbow down and wrist flipped.

The inverse kinematics can be solved as follows: First the point $\boldsymbol{p}_5$ at the origin of frame 5 is found from the end effector position and orientation from

$$\boldsymbol{p}_5 = \boldsymbol{p}_e - d_6 \boldsymbol{a}_e = \begin{bmatrix} x_5 \\ y_5 \\ z_5 \end{bmatrix} \tag{413}$$

Then joint angle 1 can be found by considering the horizontal projection of vector $\boldsymbol{p}_5$, which has the angle

$$\psi_1 = \operatorname{atan2}(y_5, x_5)$$

about the $\boldsymbol{z}_0$ axis from the $x_0$ axis. Note that if $x_5 = y_5 = 0$, then the solution is undefined. This happens for the shoulder singularity of the manipulator, which occurs when the point $\boldsymbol{p}_5$ is on the $z_0$ axis. The angle $\theta_1$ differs from $\psi_1$ due to the offset $d_4$ corresponding to an offset angle

$$\delta_1 = \operatorname{atan2}\left(d_4, \sqrt{x_5^2 + y_5^2 - d_4^2}\right).$$

This gives

$$\theta_1 = \psi_1 + \delta_1$$

Note that this is only defined when $x_5^2 + y_5^2 \geq d_4^2$, which is always satisfied due to the design of the manipulator.

Given $\theta_1$, the unit vectors of frame 1 are computed as

$$\boldsymbol{x}_1 = \begin{bmatrix} c_1 \\ s_1 \\ 0 \end{bmatrix}, \quad \boldsymbol{y}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{z}_1 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}$$

Next the vector $\boldsymbol{z}_4$ is computed. This vector is found from the observation that it is perpendicular to both $\boldsymbol{z}_1$ and $\boldsymbol{a}_e$. Care must be taken to ensure the right sign. It is seen that in the home position the vector can be computed from

$$\boldsymbol{z}_4 = \frac{\boldsymbol{z}_1 \times \boldsymbol{a}_e}{|\boldsymbol{z}_1 \times \boldsymbol{a}_e|}, \quad |\boldsymbol{z}_1 \times \boldsymbol{a}_e| \neq 0 \tag{414}$$

Note that if $\boldsymbol{z}_1 \times \boldsymbol{a}_e = 0$ the solution is undefined. This happens for the wrist singularity of the manipulator.

It is seen that $\boldsymbol{y}_4 = \boldsymbol{z}_1$. Then $\boldsymbol{x}_4$ can be computed using the right-hand rule as

$$\boldsymbol{x}_4 = \boldsymbol{z}_1 \times \boldsymbol{z}_4 \tag{415}$$

Now that unit vectors of frame 4 has been computed the wrist angles $\theta_5$ and $\theta_6$ can be found. It is noted that $\theta_5$ is the angle between $\boldsymbol{a}_e$ and $\boldsymbol{y}_4$, and is found from

$$s_5 = -\boldsymbol{a}_e \cdot \boldsymbol{x}_4, \quad c_5 = \boldsymbol{a}_e \cdot \boldsymbol{y}_4 \tag{416}$$

$$\theta_5 = \text{Atan2}(-\boldsymbol{a}_e \cdot \boldsymbol{x}_4, \boldsymbol{a}_e \cdot \boldsymbol{y}_4) \tag{417}$$

The angle $\theta_6$ is the angle between $\boldsymbol{s}_e$ and $\boldsymbol{y}_5$, and is found from $\cos\theta_6 = \boldsymbol{y}_5 \cdot \boldsymbol{s}_e$, $\sin\theta_6 = \boldsymbol{y}_5 \cdot \boldsymbol{n}_e$ and $\boldsymbol{y}_5 = -\boldsymbol{z}_4$, which gives

$$\theta_6 = \text{Atan2}(-\boldsymbol{z}_4 \cdot \boldsymbol{n}_e, -\boldsymbol{z}_4 \cdot \boldsymbol{s}_e) \tag{418}$$

The angles $\theta_2$ and $\theta_3$ are found from the position $\boldsymbol{p}_3$ of the origin of frame 3, which is computed from

$$\boldsymbol{p}_3 = \boldsymbol{p}_5 - d_5\boldsymbol{z}_4 - d_4\boldsymbol{z}_1 \tag{419}$$

The coordinates of the vector are written $\boldsymbol{p}_3 = [p_{3x}, p_{3y}, p_{3z}]^{\mathrm{T}}$. Then $\theta_2$ and $\theta_3$ is found from the law of cosines for the triangle with end point with horizontal coordinate $p_h = \sqrt{p_{3x}^2 + p_{3y}^2}$ and vertical coordinate $p_v = p_{3z} - d_1$, which gives

$$c_3 = \frac{p_h^2 + p_v^2 - a_2^2 - a_3^2}{2a_2a_3} \tag{420}$$

$$s_3 = \sqrt{1 - c_3^2} \tag{421}$$

$$\theta_3 = \text{Atan2}(s_3, c_3) \tag{422}$$

and

$$c_2 = \frac{p_h(a_2 + a_3c_3) + p_va_3s_3}{p_h^2 + p_v^2} \tag{423}$$

$$s_2 = \frac{p_v(a_2 + a_3c_3) - p_ha_3s_3}{p_h^2 + p_v^2} \tag{424}$$

$$\theta_2 = \text{Atan2}(s_2, c_2) \tag{425}$$

The last angle $\theta_4$ is found by computing $\theta_{234} = \theta_2 + \theta_3 + \theta_4$. An expression for $\theta_{234}$ is found by noting that the $\boldsymbol{z}_4$ vector is rotated by an angle $\theta_{234} = \theta_2 + \theta_3 + \theta_4$ about the $\boldsymbol{z}_1$ axis starting from the $-\boldsymbol{z}_0 = -\boldsymbol{y}_1$ axis. This gives

$$s_{234} = \boldsymbol{z}_4 \cdot \boldsymbol{x}_1, \quad c_{234} = -\boldsymbol{z}_4 \cdot \boldsymbol{y}_1 \tag{426}$$

and

$$\theta_{234} = \text{Atan2}(\boldsymbol{z}_4 \cdot \boldsymbol{x}_1, \boldsymbol{z}_4 \cdot \boldsymbol{y}_1) \tag{427}$$

and the $\theta_4$ is found from

$$\theta_4 = \theta_{234} - (\theta_2 + \theta_3) \tag{428}$$

Finally, it will be shown how the different solutions for shoulder left and right, elbow up and down and wrist flipped or not are calculated. The do this, the logical variables $S$, $E$ and $W$ are defined so that

$$S = \begin{cases} 1, & \text{shoulder left} \\ -1, & \text{should right} \end{cases} \tag{429}$$

$$E = \begin{cases} 1, & \text{elbow up} \\ -1, & \text{elbow down} \end{cases} \tag{430}$$

$$W = \begin{cases} 1, & \text{wrist not flipped} \\ -1, & \text{wrist flipped} \end{cases}, \tag{431}$$

If the shoulder is changed to right, this means that the manipulator is turned around the vertical axis and reaches backwards. Then $\psi_1 = \text{atan2}(Sy_5, Sx_5)$ and $\theta_1 = \psi_1 + S\delta_1$. Moreover, the $\boldsymbol{z}_4$ vector changes to $\boldsymbol{z}_4 = S\boldsymbol{z}_1 \times \boldsymbol{a}_e/|\boldsymbol{z}_1 \times \boldsymbol{a}_e|$, and $p_h = S\sqrt{p_{3x}^2 + p_{3y}^2}$.

In the case the elbow is changed to elbow down the solution must be changed with $s_3 = E\sqrt{1 - c_3^2}$.

If the wrist is flipped, the solution for $z_4$ must be changed to $\boldsymbol{z}_4 = WS\boldsymbol{z}_1 \times \boldsymbol{a}_e/|\boldsymbol{z}_1 \times \boldsymbol{a}_e|$.

The complete solution accounting for the logical variables $S$, $E$ and $W$ is then achieved with the modifications

$$\psi_1 = \text{atan2}(Sy_5, Sx_5) \tag{432}$$

$$\theta_1 = \psi_1 + S\delta_1 \tag{433}$$

$$\boldsymbol{z}_4 = WS\frac{\boldsymbol{z}_1 \times \boldsymbol{a}_e}{|\boldsymbol{z}_1 \times \boldsymbol{a}_e|} \tag{434}$$

$$p_h = S\sqrt{p_{3x}^2 + p_{3y}^2} \tag{435}$$

$$s_3 = E\sqrt{1 - c_3^2} \tag{436}$$

### MATLAB Example

```
% Script for calculating the inverse kinematics of the UR5
clear;
SetDH_UR5;
% Input joint angles
q = qh;
q = qh - 0.1*ones(6,1);
% q(1) = q(1) + pi;
% q(2) = q(2) - pi;
```

```
[J, T] = ForwardKinUR5(dh, q);
qa = AnalyticInvKinUR5(dh, T);
q_computed = qa'
q_actual = q'

[Ja, Ta] = ForwardKinUR5(dh, qa); % Test
dT = T - Ta% Zero matrix expected

%%%%%%%%%%%%%%%%%

function q = AnalyticInverseKinUR5(dh, Td)

Te = Td;

ne = Te(1:3,1); se = Te(1:3,2); ae = Te(1:3,3);
pe = Te(1:3,4);

d1 = dh(1,3); d4 = dh(4,3); d5 = dh(5,3); d6 = dh(6,3);
a2 = dh(2,1); a3 = dh(3,1); L2 = a2; L3 = a3;

p5 = pe - d6*ae;
p5x = p5(1); p5y = p5(2);
SHOULDER = 1;
psi5 = atan2(SHOULDER*p5y,SHOULDER*p5x);
offset1 = atan2(d4, sqrt(p5x^2 + p5y^2 - d4^2));

q1 = psi5 + SHOULDER*offset1;

x1 = [cos(q1); sin(q1); 0];
y1 = [0; 0; 1];
z1 = [sin(q1); -cos(q1); 0];

WRIST = 1;
z4 = WRIST*SHOULDER*cross(z1,ae); z4 = z4/norm(z4);
y4 = z1;
x4 = cross(y4,z4);

q5 = atan2(dot(-ae,x4),dot(ae,y4));

q6 = atan2(dot(-z4,ne), dot(-z4,se));

p3 = p5 - d5*z4 - d4*z1; p3x = p3(1); p3y = p3(2); p3z = p3(3);

pv = p3(3) - d1; ph = SHOULDER*sqrt(p3(1)^2+p3(2)^2);

q234 = atan2(dot(z4,x1),dot(z4,-y1));
```

```
ELBOW = -1;

c3 = (ph^2 + pv^2 - L2^2 - L3^2) / (2*L2*L3);
s3 = ELBOW*sqrt(1-c3^2);
q3 = atan2(s3,c3);

c2 = (ph*(L2 + L3*c3) + pv*L3*s3) / (L2^2 + L3^2 + 2*L2*L3*c3);
s2 = (pv*(L2 + L3*c3) - ph*L3*s3) / (L2^2 + L3^2 + 2*L2*L3*c3);
c2 = (ph*(L2 + L3*c3) + pv*L3*s3) / (ph^2 + pv^2);
s2 = (pv*(L2 + L3*c3) - ph*L3*s3) / (ph^2 + pv^2);
q2 = atan2(s2,c2);

q4 = q234 - (q2 + q3 );
if q4 > pi
    q4 = q4 -2*pi;
end

q = [q1; q2; q3; q4; q5; q6];

%%%%%%%%%%%%%%%%%

% Denavit Hartenberg Parameters for UR5 robot with 6 rotational joints
% [a alpha d theta]
dh = zeros(6,4); % Declaration of dh as matrix of dimension 6x4
dh(1,:) = [ 0          pi/2   0.0892  0  ];
dh(2,:) = [-0.425      0      0       0  ];
dh(3,:) = [-0.39243    0      0       0  ];
dh(4,:) = [ 0          pi/2   0.109   0  ];
dh(5,:) = [ 0         -pi/2   0.093   0  ];
dh(6,:) = [ 0          0      0.082   0  ];

% Home position
qh =[0   -pi/2   -pi/2 -pi/2   pi/2  0]';

%%%%%%%%%%%%%%%%

function [J, Te] = ForwardKinUR5(dh, q)
% dh = [a alpha d theta]

dhf = dh;
% Insert joint variable in DH vectors
dhf(:,4) = dhf(:,4) + q(:);

% Homogeneous matrices for links
T01 =  LinkDH2T (dhf(1,:));
T12 =  LinkDH2T (dhf(2,:));
```

```
T23 =  LinkDH2T (dhf(3,:));
T34 =  LinkDH2T (dhf(4,:));
T45 =  LinkDH2T (dhf(5,:));
T56 =  LinkDH2T (dhf(6,:));

% Homogeneous matrices for intermediate frames
T02 = T01 * T12;
T03 = T02 * T23;
T04 = T03 * T34;
T05 = T04 * T45;

% Homogeneous matrix for manipulator
Te = T05 * T56;

% Rotation axes
z0 = [0 0 1]';
z1 = T01(1:3,3);
z2 = T02(1:3,3);
z3 = T03(1:3,3);
z4 = T04(1:3,3);
z5 = T05(1:3,3);

% Position vectors
p0 = [0 0 0]';
p1 = T01(1:3,4);
p2 = T02(1:3,4);
p3 = T03(1:3,4);
p4 = T04(1:3,4);
p5 = T05(1:3,4);

pe = Te(1:3,4);

% Geometric Jacobian for rotational joints
J = [ cross(z0,(pe-p0)) cross(z1,(pe-p1)) cross(z2,(pe-p2)) ...
      cross(z3,(pe-p3)) cross(z4,(pe-p4)) cross(z5,(pe-p5)) ;
      z0 z1 z2 z3 z4 z5 ];
```

### 10.9 Singularities of the UR5 industrial robot

The UR5 manipulator has three singularities.

- The elbow singularity occurs when the elbow in joint 3 is stretched out. Joints 2 and 3 will in general move the wrist point in two degrees of freedom in the plane that is perpendicular to joint axes 2 and 3. In the singularity when the elbow is stretched out, the wrist point can only be moved in one degree of freedom in this plane. The condition for the elbow singularity is $q_3 = 0$.

- The shoulder singularity occurs when the distance from point $\boldsymbol{p}_5$ to the $\boldsymbol{z}_0$ axis is $d_4$. In this case the point $p_5$ cannot have a velocity in the $\boldsymbol{z}_1$ direction, which means that the range space of the Jacobian is reduced by one degree of freedom.
- The wrist singularity has the condition $q_5 = 0$. In this case the joint axes of joints 4 and 6 will be aligned, and there can be no rotation about the axis that is orthogonal to joint axes 4, 5 and 6 without contribution from axes 1, 2 and 3, which reduces the dimension of the range space of $J$. In the inverse kinematic solution the expression for $\boldsymbol{z}_4$ becomes undefined as $\boldsymbol{z}_1 \times \boldsymbol{a}_e = 0$.

## 11 Numerical optimization

Numerical optimization is used in many allocations. The techniques are based on iterative improvement by incrementing the solution based on gradient information. A basic reference on methods for numerical optimization is [12]. In robot kinematics it is used in iterative inverse kinematics, which is a solution to the inverse kinematics where the Jacobian is used in an incremental solution technique instead of the direct computation used in analytic inverse kinematics. Numerical optimization is also widely used in robot vision.

### 11.1 Linear least-squares

Consider a system of $m$ linear equations in $n$ unknowns given by

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{437}$$

Here $\boldsymbol{x}$ is a vector of $n$ unknowns, $\boldsymbol{A}$ is an $m \times n$ matrix, and $\boldsymbol{b}$ is a vector of dimension $m$. It is assumed that $m > n$ equations, which means that the system of equations is overdetermined, as there are more equations than unknown variables. This means that in general there is no solution $\boldsymbol{x}$ to the problem. In applications this is usually solved by finding the least-squares solution $\boldsymbol{x}^*$ to the problem. This is done by minimizing a quadratic objective function

$$f_L(\boldsymbol{x}) = \frac{1}{2}(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b})^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) \tag{438}$$

with respect to $\boldsymbol{x}$. The gradient of the cost function with respect to $\boldsymbol{x}$ is

$$\boldsymbol{\nabla} f_L(\boldsymbol{x}) = -\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} \tag{439}$$

where

$$\boldsymbol{\nabla} = \left[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right]^{\mathrm{T}}$$

is a column vector of differentiation operators and the gradient is the column vector

$$\boldsymbol{\nabla} f_L(\boldsymbol{x}) = \left[\frac{\partial f_L(\boldsymbol{x})}{\partial x_1}, \ldots, \frac{\partial f_L(\boldsymbol{x})}{\partial x_n}\right]^{\mathrm{T}} \tag{440}$$

The gradient must be zero at the optimal solution, and it follows from (438) that the optimal solution $\boldsymbol{x}^*$ can be found from

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}^* = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b} \tag{441}$$

which are the normal equations of the problem.

### 11.2 Damped linear least-squares

In certain applications the damped-least squares solution is computed by minimizing the modified objective function

$$f_d = \frac{1}{2}(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b})^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) + \frac{1}{2}\lambda^2 \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} \tag{442}$$

which gives a modified optimal solution $\boldsymbol{x}^*_\lambda$ that will depend on the selection of the damping factor $\lambda$.

This is called the damped least-squares problem since the magnitude of the unknown variable $\boldsymbol{x}$ enters in the objective function. The normal equations for the damped least-squares problem is seen to be

$$(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda^2\boldsymbol{I})\boldsymbol{x}^*_\lambda = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b} \tag{443}$$

The advantage of this solution is that it can be computed even when $\boldsymbol{A}$ is singular, which is not possible in the original problem since the normal equations will have a singular matrix $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$. Moreover, the damped-least squares solution is well-conditioned when $\boldsymbol{A}$ is close to being singular, whereas the solution $\boldsymbol{x}^*$ without damping can tend to infinity when $\boldsymbol{A}$ is close to being singular.

To investigate the solution of the damped least-squares solution further the singular value decomposition is used. The singular value decomposition of $\boldsymbol{A}$ is

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{T}} = \sum_{i=1}^{n} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^{\mathrm{T}} \tag{444}$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq 0$, where $r$ is the rank of $\boldsymbol{A}$, and the singular values $\sigma_{r+1}, \ldots, \sigma_n$ are zero if $r < n$.

Then

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda^2\boldsymbol{I} = \boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{T}} + \lambda^2\boldsymbol{I} = \boldsymbol{V}\boldsymbol{\Sigma}^2\boldsymbol{V}^{\mathrm{T}} + \lambda^2\boldsymbol{I} = \boldsymbol{V}(\boldsymbol{\Sigma}^2 + \lambda^2\boldsymbol{I})\boldsymbol{V}^{\mathrm{T}} \tag{445}$$

where it is used that $\boldsymbol{U}\boldsymbol{U}^{\mathrm{T}} = \boldsymbol{I}$ and $\boldsymbol{V}\boldsymbol{V}^{\mathrm{T}} = \boldsymbol{I}$. This can also be written

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda^2\boldsymbol{I} = \sum_{i=1}^{n}(\sigma_i^2 + \lambda^2)\boldsymbol{v}_i\boldsymbol{v}_i^{\mathrm{T}} \tag{446}$$

The inverse is

$$(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda^2\boldsymbol{I})^{-1} = \boldsymbol{V}(\boldsymbol{\Sigma}^{-2} + \lambda^{-2}\boldsymbol{I})\boldsymbol{V}^{\mathrm{T}} = \sum_{i=1}^{n}\frac{1}{\sigma_i^2 + \lambda^2}\boldsymbol{v}_i\boldsymbol{v}_i^{\mathrm{T}} \tag{447}$$

The solution to the normal equation is then

$$\boldsymbol{x}^*_\lambda = (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda^2\boldsymbol{I})^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{V}(\boldsymbol{\Sigma}^{-2} + \lambda^{-2}\boldsymbol{I})\boldsymbol{\Sigma}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{b} \tag{448}$$

which can be written

$$\boldsymbol{x}^*_\lambda = \sum_{i=1}^{n}\frac{\sigma_i}{\sigma_i^2 + \lambda^2}\boldsymbol{v}_i\boldsymbol{u}_i^{\mathrm{T}}\boldsymbol{b} \tag{449}$$

It is noted that in the original problem without damping the optimal solution is

$$\boldsymbol{x}^* = \sum_{i=1}^{n} \frac{1}{\sigma_i} \boldsymbol{v}_i \boldsymbol{u}_i^{\mathrm{T}} \boldsymbol{b} \tag{450}$$

Here $1/\sigma_n$ will tend to infinity when $\sigma_n$ tends to zero. The solution is undefined when $\sigma_n = 0$.

In the damped solution $\boldsymbol{x}_\lambda^*$ the potentially problematic factor $1/\sigma_n$ is repalced with the factor $g(\sigma_n) = \sigma_n/(\sigma_n^2 + \lambda^2)$. This factor will be well-behaved even when $\sigma_n$ tends to zero. When $\sigma_n \gg \lambda$ the solution $\boldsymbol{x}_\lambda^* \approx \boldsymbol{x}^*$ since

$$\frac{\sigma_n}{\sigma_n^2 + \lambda^2} \approx \frac{1}{\sigma_n}, \quad \text{when} \quad \sigma_n \gg \lambda \tag{451}$$

Moreover,

$$\frac{\sigma_n}{\sigma_n^2 + \lambda^2} \approx \frac{\sigma_n}{\lambda^2}, \quad \text{when} \quad \sigma_n \ll \lambda \tag{452}$$

The maximum value for $g(\sigma_n)$ is found from

$$\frac{\mathrm{d}g(\sigma_n)}{\mathrm{d}\sigma_n} = \frac{\lambda^2 - \sigma_n^2}{(\sigma_i^2 + \lambda^2)^2} = 0 \tag{453}$$

which gives the maximum value $g(\lambda) = 1/(2\lambda)$ at $\sigma_n = \lambda$.

### 11.3 Nonlinear least-squares optimization

In a nonlinear least-squares problem a solution is sought that will minimize a nonlinear objective function of the form

$$f(x) = \frac{1}{2} \sum_{i=1}^{N} r_i(x)^2 \tag{454}$$

with respect to $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$. The functions $r_i(x)$ are called the residuals of the problem. The residuals can be arranged in a residual vector

$$\boldsymbol{r}(\boldsymbol{x}) = [r_1(\boldsymbol{x}), r_2(\boldsymbol{x}), \ldots, r_N(\boldsymbol{x})]^{\mathrm{T}} \tag{455}$$

A nonlinear least-squares solution will in general not have a closed-form solution. Instead the solution must be found by iteration using a numerical optimization method. This is done in an iterative scheme where the solution at step $k+1$ is found from the solution at step $k$ by the computation

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{p}_k \tag{456}$$

where $\boldsymbol{p}_k$ is the increment of the solution, which is calculated from the first and second order derivatives of the objective function $f(\boldsymbol{x})$. The first order derivative, which is the gradient, is written in terms of the column vector

$$\boldsymbol{\nabla} = \left[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right]^{\mathrm{T}}$$

of differentiation operators. Then the gradient of a residual $r_i$ is

$$\boldsymbol{\nabla} r_i(\boldsymbol{x}) = \left[\frac{\partial r_i}{\partial x_1}, \ldots, \frac{\partial r_i}{\partial x_n}\right]^{\mathrm{T}}$$

The gradient of the cost function can be written in either of the two forms

$$\boldsymbol{\nabla} f(\boldsymbol{x}) = \sum_{i=1}^{N} r_i \boldsymbol{\nabla} r_i = \sum_{i=1}^{N} (\boldsymbol{\nabla}^{\mathrm{T}} r_i) r_i$$

The notation $\boldsymbol{\nabla}^2 = \boldsymbol{\nabla}\boldsymbol{\nabla}^{\mathrm{T}}$ is commonly used for the second order derivative, which is the Hessian

$$\boldsymbol{\nabla}^2 f(x) = \boldsymbol{\nabla}\boldsymbol{\nabla}^{\mathrm{T}} f(x) = \left\{\frac{\partial^2 f(\boldsymbol{x})}{\partial x_i \partial x_j}\right\} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix} \tag{457}$$

where the notation $\boldsymbol{\nabla}^2 = \boldsymbol{\nabla}\boldsymbol{\nabla}^{\mathrm{T}}$ is commonly used. The Hessian is found to be

$$\boldsymbol{\nabla}^2 f(x) = \boldsymbol{\nabla}\boldsymbol{\nabla}^{\mathrm{T}} f(x) = \sum_{i=1}^{N} \boldsymbol{\nabla}(r_i \boldsymbol{\nabla}^{\mathrm{T}} r_i) = \sum_{i=1}^{N} \boldsymbol{\nabla} r_i \boldsymbol{\nabla}^{\mathrm{T}} r_i + \sum_{i=1}^{N} r_i \boldsymbol{\nabla}^2 r_i$$

The terms $\boldsymbol{\nabla}^2 r_i$ are somewhat complicated to compute, and they appear in multiplication with $r_i$, which will be close to zero when $\boldsymbol{x}$ approaches the optimal solution. Therefore, the Hessian will often be approximated by

$$\boldsymbol{\nabla}^2 f(x) \approx \sum_{i=1}^{N} \boldsymbol{\nabla} r_i \boldsymbol{\nabla}^{\mathrm{T}} r_i \tag{458}$$

This is easy to compute, which will be seen in the following.

The Jacobian of the problem is the matrix of first-order derivatives of the residuals, that is,

$$\boldsymbol{J}(\boldsymbol{x}) = \left\{\frac{\partial r_i}{\partial x_j}\right\} = \begin{bmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_N}{\partial x_1} & \cdots & \frac{\partial r_N}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\nabla}^{\mathrm{T}} r_1 \\ \vdots \\ \boldsymbol{\nabla}^{\mathrm{T}} r_N \end{bmatrix} \tag{459}$$

The gradient of the objective function can be expressed in terms of the Jacobian as

$$\boldsymbol{\nabla} f(\boldsymbol{x}) = \boldsymbol{J}(\boldsymbol{x})^{\mathrm{T}} \boldsymbol{r}(x) \tag{460}$$

Moreover, the Hessian can be written in terms of the Jacobian as

$$\boldsymbol{\nabla}^2 f(\boldsymbol{x}) \approx \sum_{i=1}^{N} \boldsymbol{\nabla} r_i \boldsymbol{\nabla}^{\mathrm{T}} r_i = \boldsymbol{J}(\boldsymbol{x})^{\mathrm{T}} \boldsymbol{J}(\boldsymbol{x}) \tag{461}$$

where the second order terms $\boldsymbol{\nabla}^2 r_i$ has been left out.

The nonlinear least-squares optimization problem is solved by defining a linear least-squares problem at each iteration $k$. This is done by defining a quadratic approximation of the objective function based on the Taylor series expansion of the objective function about $\boldsymbol{x}_k$, which is

$$f(\boldsymbol{x}_k + \boldsymbol{p}) = f(\boldsymbol{x}_k) + \boldsymbol{\nabla}^{\mathrm{T}} f(\boldsymbol{x}_k)\boldsymbol{p} + \frac{1}{2}\boldsymbol{p}_k^{\mathrm{T}} \boldsymbol{\nabla}^2 f(x_k)\boldsymbol{p} + \dots \tag{462}$$

Here $\boldsymbol{p}$ is the increment from $\boldsymbol{x}_k$. Typically, the quadratic approximation of the objective function at step $k$ will be selected as

$$m_k(\boldsymbol{p}) = f(\boldsymbol{x}_k) + \boldsymbol{r}^{\mathrm{T}}\boldsymbol{J}(\boldsymbol{x}_k)\boldsymbol{p} + \frac{1}{2}\boldsymbol{p}^{\mathrm{T}}\boldsymbol{J}(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{J}(\boldsymbol{x}_k)\boldsymbol{p} \tag{463}$$

which is optimized with respect to $\boldsymbol{p}$ in a linear least-squares problem. The optimal solution for $\boldsymbol{p}$ is found by setting the gradient of $\boldsymbol{m}_k(\boldsymbol{p})$ to zero. From

$$\boldsymbol{\nabla} m_k(\boldsymbol{p}) = \boldsymbol{J}(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{r} + \boldsymbol{J}(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{J}(\boldsymbol{x}_k)\boldsymbol{p} = \boldsymbol{0} \tag{464}$$

The step $\boldsymbol{p}_k$ is calculated as the optimal solution of this equation, which is found from the normal equations

$$\boldsymbol{J}(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{J}(\boldsymbol{x})\boldsymbol{p}_k = -\boldsymbol{J}(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{r} \tag{465}$$

The solution at step $k+1$ is the calculated from the update equation

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{p}_k \tag{466}$$

The optimization is then repeated at $\boldsymbol{x}_{k+1}$ as a linear least-squares problem with objective function $m_{k+1}(\boldsymbol{p}) = f(\boldsymbol{x}_{k+1}) + \boldsymbol{r}^{\mathrm{T}}\boldsymbol{J}(\boldsymbol{x}_{k+1})\boldsymbol{p} + \frac{1}{2}\boldsymbol{p}^{\mathrm{T}}\boldsymbol{J}(\boldsymbol{x}_{k+1})^{\mathrm{T}}\boldsymbol{J}(\boldsymbol{x}_{k+1})\boldsymbol{p}$ to calculate $\boldsymbol{x}_{k+2}$. This iteration is continued until the solution has converged according to some convergence criteria like

$$\frac{\|\boldsymbol{p}_{k+1} - \boldsymbol{p}_k\|}{\|\boldsymbol{p}_k\|} < 0.001 \tag{467}$$

or

$$\frac{\|\boldsymbol{f}(\boldsymbol{x}_{k+1}) - \boldsymbol{f}(\boldsymbol{x}_k)\|}{\|\boldsymbol{f}(\boldsymbol{x}_k)\|} < 0.0001 \tag{468}$$

## 12 Iterative inverse kinematics

The inverse kinematics problem is the problem of calculating the joint vector when the position and orientation of the end-effector is given. This is widely used in applications where the position and orientation of the end effector can be specified to perform a task like gripping an object, or performing a spot welding operation. The inverse kinematics can be done analytically where the joint angles are calculated from the end effector position and orientation using geometric relations. The is possible for certain types of six-joint manipulators, like the anthropomorphic manipulator with a spherical wrist joint, or manipulators where joint axes 2, 3 and 4 are parallel. Alternatively, the inverse kinematics can be solved iteratively using the manipulator Jacobian.

### 12.1 Resolved motion rate control

Resolved rate control was proposed already in 1969 by Dan Whitney [21], primarily for use in teleoperation, where an operator controls a manipulator in a master-slave arrangement or with a joy-stick. In a master-slave arrangement the operator moves a master manipulator, and the slave manipulator follows the motions of the master. This was used for handling hazardous material in connection with research and experiments of radioactive material, and for handling of biological experiments. In the first installations of such systems the operator would watch the slave manipulator through a protective window, and control the handling of an experiment with a master manipulator. Later this was rearranged for remote operation where the operator watched the slave on a video screen, and the signals between the master ans the slave manipulators was transmitted a over a communication channel.

The idea of resolved motion rate control was to control the motion of the end effector of the remote manipulator by commanding the velocity and the angular velocity of the end effector, and then compute the required joint rates. Then it was not required to have a master manipulator to command the velocities. Let the joint variables of the remote manipulator be given by $\boldsymbol{q}$, let the velocity of the remote end-effector be $\boldsymbol{v}$, and let the angular velocity be $\boldsymbol{\omega}$. Then

$$\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{469}$$

where $\boldsymbol{J}(\boldsymbol{q})$ is the geometric Jacobian. Given the commanded velocities $\boldsymbol{v}$ and $\boldsymbol{\omega}$ from the operator, the required joint velocities are found from the resolved motion rate control

$$\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})^{-1} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} \tag{470}$$

Since the velocity is controlled, there is a need to correct from information on position and rotation of the end effector. In this scheme this is done by the operator, who watches the end effector through a protective glass, or on a monitor screen.

### 12.2 Inverse kinematics

The inverse kinematic problem is the computation of the joint vector $\boldsymbol{q}$ corresponding to a desired configuration given by

$$\boldsymbol{T}_d = \begin{bmatrix} \boldsymbol{R}_d & \boldsymbol{p}_d \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix} \tag{471}$$

where $\boldsymbol{R}_d$ is the desired rotation matrix, and $\boldsymbol{p}_d$ is the desired position vector.

There are two methods for solving this problem. One is the analytic approach, where geometry is used to derive equations that are used to solve for the unknown $\boldsymbol{q}$. These equations are in general nonlinear, and there will in general be several solutions. Analytic solutions are available for manipulators that satisfy certain criteria on the geometry. One type of manipulators that have analytic solutions is manipulators where there are three intersection wrist axes, which is the case for many industrial manipulators. The other type of manipulators that have an analytic solution for the inverse kinematics is manipulators where joint axes 2, 3 and 4 are parallel, like the UR5 and the Cincinnati Milacron T3.

The other method for computing the inverse kinematics is the iterative approach, where the inverse Jacobian of the manipulator is used in an iterative solution. This approach can be applied for all serial-link manipulators. Both the geometric Jacobian and the analytic Jacobian can be used in the iterative inverse kinematic solution.

### 12.3 Inverse kinematics using the analytic Jacobian

Suppose that the desired position and orientation of the end effector is given by

$$\boldsymbol{x}_d = \begin{bmatrix} \boldsymbol{p}_d \\ \boldsymbol{\phi}_d \end{bmatrix} \tag{472}$$

where the rotation is described by the vector $\boldsymbol{\phi}_d$ with 3 elements. The deviation in the position and orientation vector is

$$\boldsymbol{e} = \boldsymbol{x}_d - \boldsymbol{x}_e = \boldsymbol{x}_d - \boldsymbol{h}(\boldsymbol{q}) \tag{473}$$

Separation of the position and orientation part gives

$$\begin{bmatrix} \boldsymbol{e}_p \\ \boldsymbol{e}_o \end{bmatrix} = \begin{bmatrix} \boldsymbol{p}_d - \boldsymbol{p}_e \\ \boldsymbol{\phi}_d - \boldsymbol{\phi}_e \end{bmatrix} = \begin{bmatrix} \boldsymbol{p}_d - \boldsymbol{h}_e(\boldsymbol{q}) \\ \boldsymbol{\phi}_d - \boldsymbol{h}_o(\boldsymbol{q}) \end{bmatrix} \tag{474}$$

The joint vector $\boldsymbol{q}_d$ corresponding to the desired configuration $\boldsymbol{x}_d = \boldsymbol{h}(\boldsymbol{q}_d)$ can then be found by nonlinear least-squares problem with objective function

$$f(\boldsymbol{q}) = \frac{1}{2}[\boldsymbol{h}(\boldsymbol{q}) - \boldsymbol{x}_d]^{\mathrm{T}}[\boldsymbol{h}(\boldsymbol{q}) - \boldsymbol{x}_d] \tag{475}$$

where $\boldsymbol{r} = \boldsymbol{h}(\boldsymbol{q}) - \boldsymbol{x}_d$ is the residual vector. The Jacobian of this nonlinear least-squares problem is

$$\boldsymbol{J}_A = \left\{ \frac{\partial h_i}{\partial q_j} \right\} \tag{476}$$

which is the analytic Jacobian of the manipulator. The normal equation to be optimized at each iteration is therefore

$$\boldsymbol{J}_A^{\mathrm{T}}(\boldsymbol{q}_k)\boldsymbol{J}_A(\boldsymbol{q}_k)\Delta\boldsymbol{q} = \boldsymbol{J}_A(\boldsymbol{q}_k)^{\mathrm{T}}[\boldsymbol{x}_d - \boldsymbol{h}(\boldsymbol{q}_k)] \tag{477}$$

When the manipulator is not in a singular configuration the Jacobian will be invertible, and the optimal step is found to be

$$\Delta\boldsymbol{q}_k^* = \boldsymbol{J}_A(\boldsymbol{q}_k)^{-1}[\boldsymbol{x}_d - \boldsymbol{h}(\boldsymbol{q}_k)] \tag{478}$$

This is repeated until the deviation $\boldsymbol{x}_d - \boldsymbol{h}(\boldsymbol{q}_k)$ is sufficiently small. This method converges fast, and if the starting point is close to the correct value, the method will converge reasonably well in one iteration.

The algorithm is:

- Input tolerance $\delta > 0$, $\boldsymbol{x}_d$ and the initial value for $\boldsymbol{q}$
- while $\|\boldsymbol{e}\| > \delta$
  - ⋆ Calculate $\boldsymbol{h}(\boldsymbol{q})$ and $\boldsymbol{J}_A(\boldsymbol{q})$.

  - ⋆ Calculate the error $\boldsymbol{e} = \boldsymbol{x}_d - \boldsymbol{h}(\boldsymbol{q})$
  - ⋆ Update $\boldsymbol{q} := \boldsymbol{q} + \boldsymbol{J}_a(\boldsymbol{q})^{-1}[\boldsymbol{x}_d - \boldsymbol{h}(\boldsymbol{q})]$
- end

In the case that the manipulator is moving along a trajectory given by a desired position and velocity $\boldsymbol{x}_d(t)$, it may be sufficient with one iteration per time-step $t_k$.

An intuitive explanation of the update equation can be made by observing that $\dot{\boldsymbol{x}} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}$ implies that for small steps the increment in $\boldsymbol{x}$ can be related to an increment i $\boldsymbol{q}$ as $\Delta\boldsymbol{x} = \boldsymbol{J}(\boldsymbol{q})\Delta\boldsymbol{q}$, which gives $\Delta\boldsymbol{q} \approx \boldsymbol{J}(\boldsymbol{q})^{-1}\Delta\boldsymbol{x}$. Therefore, an increment $\Delta\boldsymbol{x} = \boldsymbol{x}_d - \boldsymbol{h}(\boldsymbol{q})$ that will bring the manipulator to the desired position and orientation given by $\boldsymbol{x}_d$, can be approximately achieved with the increment $\Delta\boldsymbol{q} = \boldsymbol{J}_a(\boldsymbol{q})^{-1}[\boldsymbol{x}_d - \boldsymbol{h}(\boldsymbol{q})]$. By repeating this correction the error is $\boldsymbol{x}$ can be made sufficiently small for a given accuracy requirement.

### 12.4 Inverse kinematics using the geometric Jacobian

Iterative inverse kinematics can be solved in terms of the geometric Jacobian. In that case the error vector can be written in the form

$$\boldsymbol{e} = \begin{bmatrix} \boldsymbol{e}_p \\ \boldsymbol{e}_o \end{bmatrix} \tag{479}$$

where

$$\boldsymbol{e}_p = \boldsymbol{p}_d - \boldsymbol{p}_e = \boldsymbol{p}_d - \boldsymbol{h}_p(\boldsymbol{q}) \tag{480}$$

is the error in position, and the vector $\boldsymbol{e}_o$, which is of dimension 3 is the error in orientation. In this case there is no 3-dimensional vector that describes the orientation of the end effector, and the situation is therefore different from the solution using the analytic Jacobian where the deviation in orientation was given in terms of the Euler angles.

The deviation in orientation is described by

$$\boldsymbol{R} = \boldsymbol{R}_d\boldsymbol{R}_e^{\mathrm{T}} \tag{481}$$

which is a result of describing the deviation from $\boldsymbol{R}_e$ to $\boldsymbol{R}_d$ by $\boldsymbol{R}_d = \boldsymbol{R}\boldsymbol{R}_e$. Let the angle-axis parameters for $\boldsymbol{R}$ be $\theta$ and $\boldsymbol{k}$. Then $\boldsymbol{R}$ is given by $\boldsymbol{R} = \boldsymbol{I} + \sin\theta\boldsymbol{k}^\times + (1-\cos\theta)\boldsymbol{k}^\times\boldsymbol{k}^\times$, and the deviation from $\boldsymbol{R}_e$ to $\boldsymbol{R}_d$ is described by a rotation by an angle $\theta$ about the axis given by the unit vector $\boldsymbol{k}$. The usual solution is to describe the error in orientation by the Euler rotation vector $\boldsymbol{e}_o = \sin\theta\boldsymbol{k}$. This vector is easily computed, and has the required characteristic of giving a step in the right direction.

It is noted that $\boldsymbol{\omega} = \boldsymbol{J}_o(\boldsymbol{q})\dot{\boldsymbol{q}}$. This means that an increment $\alpha\boldsymbol{e}_o$ in rotation can be achieved with the angular velocity $\boldsymbol{\omega} = \omega\boldsymbol{k}$ over a time interval of $\Delta t$ where $\Delta t\omega = \alpha\sin\theta = \|\alpha\boldsymbol{e}_o\|$. The corresponding increment in $\boldsymbol{q}$ is $\Delta\boldsymbol{q} \approx \dot{\boldsymbol{q}}\Delta t$, which therefore satisfies $\alpha\boldsymbol{e}_o \approx \boldsymbol{J}_o(\boldsymbol{q})\Delta\boldsymbol{q}$.

Combining this with the translational part, it is seen that an increment in $\boldsymbol{q}$ can be related to the deviations $\boldsymbol{e}_p$ and $\boldsymbol{e}_o$ by

$$\begin{bmatrix} \boldsymbol{e}_p \\ \boldsymbol{e}_o \end{bmatrix} = \begin{bmatrix} \boldsymbol{J}_p(\boldsymbol{q}) \\ \boldsymbol{J}_o(\boldsymbol{q}) \end{bmatrix} \Delta\boldsymbol{q} = \boldsymbol{J}(\boldsymbol{q})\Delta\boldsymbol{q} \tag{482}$$

and the increment is $\boldsymbol{q}$ can therefore be found from

$$\Delta \boldsymbol{q} = \boldsymbol{J}(\boldsymbol{q})^{-1}\boldsymbol{e} \tag{483}$$

An iterative scheme can therefore be implemented with the algorithm:

- Input tolerance $\delta > 0$, $\boldsymbol{T}_d$ and the initial value for $\boldsymbol{q}$
- while $\|\boldsymbol{e}\| > \delta$
  - ⋆ Calculate $\boldsymbol{T}_e(\boldsymbol{q})$ and $\boldsymbol{J}(\boldsymbol{q})$.
  - ⋆ Calculate the error in terms of $\boldsymbol{e}_p$ and $\boldsymbol{e}_o$ from $\boldsymbol{T}_e$ and $\boldsymbol{T}_d$.
  - ⋆ Update $\boldsymbol{q} := \boldsymbol{q} + \boldsymbol{J}(\boldsymbol{q})^{-1}\boldsymbol{e}$
- end

### 12.5 Combining resolved rate and iterative inverse kinematics

A manipulator that is moving along a desired trajectory defined by $\boldsymbol{T}_d(t) \in SE(3)$ and its time derivative $\dot{\boldsymbol{T}}_d(t)$ will have a desired position and orientation for the end effector that is depending on time $t$. The desired velocity $\boldsymbol{v}_d(t)$ and $\boldsymbol{\omega}_d(t)$ will also be time dependent and depending on $\dot{\boldsymbol{T}}_d$. The corresponding desired joint velocity vector will be given by

$$\dot{\boldsymbol{q}}_d = \boldsymbol{J}(\boldsymbol{q})^{-1}\begin{bmatrix} \boldsymbol{v}_d \\ \boldsymbol{\omega}_d \end{bmatrix} \tag{484}$$

Suppose that the desired trajectory is defined at a sampling interval $h$, which means that the trajectory is given at discrete time instants $t_k = t_0 + kh$, $k = 0, 1, 2, \ldots, N$ with an interval $h$ between $t_k$ and $t_{k+1}$. Then the resolved motion rate control gives the joint velocity

$$\dot{\boldsymbol{q}}_d(t_k) = \boldsymbol{J}(\boldsymbol{q}(t_k))^{-1}\begin{bmatrix} \boldsymbol{v}_d(t_k) \\ \boldsymbol{\omega}_d(t_k) \end{bmatrix} \tag{485}$$

at time $t_k$. If the manipulator starts at $t = t_0$ with the correct joint position $\boldsymbol{q}(t_k) = \boldsymbol{q}_d(t_k)$, then it will approximately follow the trajectory when the resolved motion rate control is used, but it must be expected that some deviation $\Delta\boldsymbol{q}(t_k) = \boldsymbol{q}_d(t_k) - \boldsymbol{q}(t_k)$ will develop as time increases. Moreover, if there is a initial nonzero deviation at $t = t_0$ there is a need to correct the motion of the manipulator to ensure that is converges to the desired trajectory.

The solution is to use resolved motion rate control in addition to iterative inverse kinematics, where the initial value of the joint angles a time $t_{k+1}$ is calculated from the joint position at time $t_k$ using resolved motion rate control, and then the iterated inverse kinematic is used to correct the joint position at $t_{k+1}$.

The algorithm is:

- Input tolerance $\delta > 0$, $\boldsymbol{T}_d(t_k)$, $k = 0, 1, 1, \ldots, N$ and the initial value for $\boldsymbol{q}$
- for $k = 1, \ldots, N$
  - ⋆ $t_k = t_{k-1} + h$, $\boldsymbol{q} := \boldsymbol{q} + h\boldsymbol{J}(\boldsymbol{q})^{-1}[\boldsymbol{v}_d^{\mathrm{T}}, \boldsymbol{\omega}_d^{\mathrm{T}}]^{\mathrm{T}}$

- ⋆ while $\|\boldsymbol{e}\| > \delta$
  - ∗ Calculate $\boldsymbol{T}_e(\boldsymbol{q})$ and $\boldsymbol{J}(\boldsymbol{q})$.
  - ∗ Calculate the errors $\boldsymbol{e}_p$ and $\boldsymbol{e}_o$ from $\boldsymbol{T}_e$ and $\boldsymbol{T}_d(t_k)$.
  - ∗ Update $\boldsymbol{q} := \boldsymbol{q} + \boldsymbol{J}(\boldsymbol{q})^{-1}\boldsymbol{e}$
  - ∗ end
- ⋆ end

- end

When the manipulator is following a trajectory, and the initial configuration $\boldsymbol{q}$ is computed so that the initial deviation $\boldsymbol{e}$ is small, it is reasonable to assume that the deviations will be sufficiently small so that the iterative scheme will converge in one iteration.

### 12.6 Damped least-squares in inverse kinematics at singularities

The inverse Jacobian matrix is used in the iterative methods for inverse kinematics. A problem with this is that the Jacobian will be singular in certain geometric configurations. As an example, many industrial robot have a so-called spherical wrist where the joint axes 4, 5 and 6 intersect, and the joint angles are the $ZYZ$ Euler angles. Then the Jacobian will be singular when joint axes 4 and 6 are parallel, which occurs in the middle of the working area with $\theta_5 = 0$. In the singularity the inverse Jacobian will be undefined, and the inverse kinematics based on the use of the inverse Jacobian cannot be used as there will be a division by zero in the solution. Moreover, close to a singularity, the magnification related to the division of a number that is close to zero will give joint rates that will tend to infinity, which in practice will cause the safety systems of the robot system to switch off the robot system.

Consider a 6-link manipulator. Let the singular value decomposition of the Jacobian be given by

$$\boldsymbol{J} = \sum_{i=1}^{6} \sigma_i \boldsymbol{u}_i \boldsymbol{v}^{\mathrm{T}} \tag{486}$$

Then the inverse Jacobian will be

$$\boldsymbol{J}^{-1} = \sum_{i=1}^{6} \frac{1}{\sigma_i} \boldsymbol{v}_i \boldsymbol{u}^{\mathrm{T}} \tag{487}$$

In the damped least-squares solution the inverse Jacobian $\boldsymbol{J}^{-1}$ is replaced with the damped least-squares inverse [2]

$$\boldsymbol{J}^* = (\boldsymbol{J}^{\mathrm{T}}\boldsymbol{J} + \lambda^2 \boldsymbol{I})^{-1}\boldsymbol{J}^{\mathrm{T}} \tag{488}$$

which has singular value decomposition

$$\boldsymbol{J}^* = \sum_{i=1}^{6} \frac{\sigma_i}{\sigma_i^2 + \lambda^2} \boldsymbol{v}_i \boldsymbol{u}^{\mathrm{T}} \tag{489}$$

It is noted that $\boldsymbol{J}^{-1}$ and $\boldsymbol{J}^{*}$ has the same vectors $\boldsymbol{v}_i$ and $\boldsymbol{u}_i$, $i = 1, \ldots, 6$. The difference is in the amplification factors which are

$$\frac{\sigma_i}{\sigma_i^2 + \lambda^2} \approx \frac{\sigma_i}{\lambda^2} \quad \text{when} \quad \sigma_i \ll \lambda \tag{490}$$

while

$$\frac{\sigma_i}{\sigma_i^2 + \lambda^2} \approx \sigma_i \quad \text{when} \quad \sigma_i \gg \lambda \tag{491}$$

# 13 Screws

## 13.1 Introduction

Screws provide a powerful mathematical tool for describing important geometric objects. This includes the description of lines, which is very elegant in terms of screws. Moreover, the resultant force and torque acting on a rigid body can be described with a special type of screw called a wrench, while the velocity and angular velocity of a rigid body can be described in terms of a twist, which is also a screw. Screws can be described in a coordinate-free form, or as coordinate vectors. Screws has certain transformation rules related to rotation and change of reference.

## 13.2 Lines in 3D

Before the definition of a screw is presented it is useful to present the geometry of a line in 3D. A line in 3D will have 4 degrees of freedom, which means that it can be described in terms of 4 parameters. This is explained in [5] by considering a line and its intersection with two planes, which can be the $xy$ and the $xz$ plane. The line will be uniquely determined by the two intersection points, where each intersection point is given by two parameters. Therefore the line is can be determined with 4 parameters. Now, this example can be used to explain that a line has 4 degrees of freedom, still this is not an efficient way of describing a line, as the description will be undefined if the line is parallel to the $x$ axis, so some other description should be used.

A line can be described by the positions $\vec{p}$ and $\vec{q}$ of two points on the line, which is a 6-parameter description. Then the position of all points on the line and a direction vector $\vec{a}$ along the line. Then position of all points on the line can be given as $\lambda\vec{p} + (1 - \lambda)\vec{q}$. A direction vector $\vec{a}$ of the line can be defined as

$$\vec{a} = \vec{p} - \vec{q} \tag{492}$$

Then the equation for any point on the line can be written$\vec{q} + \lambda\vec{a}$. This shows that a line can be described by a point $\vec{q}$ on the line and a direction vector $\vec{a}$. This description has 6 parameters.

The screw description of the line is derived on the description of a line in terms of a point on the line and a direction vector. This is done by defining the moment

$$\vec{m} = \vec{p} \times \vec{a}$$

of the direction vector with reference to the origin of some reference frame. Here $\vec{p}$ is the vector from the origin of a reference frame to a point on the line. Then, if $\vec{q}$ is another point on the line, it is convenient to define $\vec{q}$ as the point on the line $\vec{m} = \vec{p} \times \vec{a} = (\vec{q} + \vec{a}) \times \vec{a} = \vec{q} \times \vec{a}$.

Next, it is assumed that the direction vector is a unit vector so that $|\vec{a}| = 1$. moreover, it is assumed that $\vec{q}$ be the point on the line that is closest to the origin. Then the moment is given by

$$\vec{m} = \vec{q} \times \vec{a} \tag{493}$$

where the three vectors are orthogonal, and therefore $\vec{a} \times \vec{m} = \vec{0}$. In addition it follows from $|\vec{a}| = 1$ that $|\vec{m}| = |\vec{q}|$, and that the vector $\vec{q}$ can be found from

$$\vec{q} = \vec{a} \times \vec{m} \tag{494}$$

This shows that if $\vec{a}$ and $\vec{m}$ is given, then the point $\vec{q}$ on the line can be found.

It has then been established that a line can be described in terms of the direction vector $\vec{a}$, which is a unit vector, and the moment $\vec{m}$. This is a description in terms of 6 parameters with the two conditions $|\vec{a}| = 1$ and $\vec{a} \times \vec{m} = \vec{0}$, which is in agreement with the fact that the line has 4 degrees of freedom. In the following it will be shown that the two vectors *avec* and *mvec* can be represented in terms of a screw describing the line.

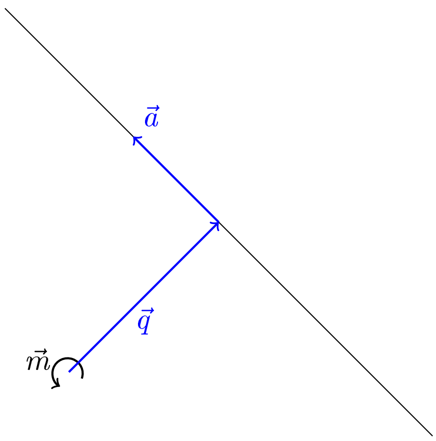



Figure 23: The direction vector $\vec{a}$ and moment $\vec{m} = \vec{q} \times \vec{a}$ of a line.

### 13.3 Definition of a screw

A screw is a pair of vectors and a screw transformation rule. A screw $\vec{s}$ is written

$$\vec{s} = (\vec{u}, \vec{v}) \tag{495}$$

where $\vec{u}$ and $\vec{v}$ are vectors characterized by their magnitude and direction.

A screw is referenced to a reference point, which could be the origin of a frame. Let $\vec{s}_a$ be a screw that is referenced to the origin of a frame denoted by $a$, and let $\vec{s}_b$ be the same screw referenced to the origin of the $b$ frame. Then the screw will satisfy the screw transformation rule

$$\vec{s}_a = (\vec{u}, \vec{v}), \quad \vec{s}_b = (\vec{u}, \vec{v} + \vec{p}_{ab} \times \vec{u}) \tag{496}$$

where $\vec{p}_{ab}$ is the vector from the origin of $a$ to the origin of $b$.

Computation on screws satisfies the following rules.

Multiplication with a scalar $\alpha$ is done by

$$\alpha\vec{s} = (\alpha\vec{u}, \alpha\vec{v}) \tag{497}$$

Addition and subtraction of screws is done component-wise. Let $\vec{s}_1 = (\vec{u}_1, \vec{v}_1)$ and $\vec{s}_2 = (\vec{u}_2, \vec{v}_2)$ be two screws. Then

$$\vec{s}_1 + \vec{s}_2 = (\vec{u}_1 + \vec{u}_2, \vec{v}_1 + \vec{v}_2) \tag{498}$$

$$\vec{s}_1 - \vec{s}_2 = (\vec{u}_1 - \vec{u}_2, \vec{v}_1 - \vec{v}_2) \tag{499}$$

The scalar product and the cross product is given by

$$\vec{s}_1 \cdot \vec{s}_2 = (\vec{u}_1 \cdot \vec{u}_2, \vec{u}_1 \cdot \vec{v}_2 + \vec{u}_2 \cdot \vec{v}_1) \tag{500}$$

$$\vec{s}_1 \times \vec{s}_2 = (\vec{u}_1 \times \vec{u}_2, \vec{u}_1 \times \vec{v}_2 + \vec{u}_2 \times \vec{v}_1) \tag{501}$$

### 13.4 Coordinate representation of screws

A screw $\vec{s} = (\vec{u}, \vec{v})$ can be represented in coordinate form as

$$\mathsf{S} = \left\{ \begin{array}{c} \boldsymbol{u} \\ \boldsymbol{v} \end{array} \right\}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are 3-dimensional column vectors. Curly braces are used to show that this is not an ordinary vector but a screw, which is defined with certain calculation rules.

A screw $\mathsf{S}_c$ with reference point $c$ is transformed to a screw $\mathsf{S}_d$ with reference point $d$ by

$$\mathsf{S}_d = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{p}_{dc}^{\times} & \boldsymbol{I} \end{bmatrix} \mathsf{S}_b \tag{502}$$

when the vectors are given in the coordinates of the same frame.

The product of a scalar $\alpha$ and a screw $\mathsf{S}$ is

$$\alpha\mathsf{S} = \left\{ \begin{array}{c} \alpha\boldsymbol{u} \\ \alpha\boldsymbol{v} \end{array} \right\}$$

The sum and difference of two screws are component-wise, that is,

$$\left\{ \begin{array}{c} \boldsymbol{u}_1 \\ \boldsymbol{v}_1 \end{array} \right\} + \left\{ \begin{array}{c} \boldsymbol{u}_2 \\ \boldsymbol{v}_2 \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{u}_1 + \boldsymbol{u}_2 \\ \boldsymbol{v}_1 + \boldsymbol{v}_2 \end{array} \right\} \tag{503}$$

$$\left\{ \begin{array}{c} \boldsymbol{u}_1 \\ \boldsymbol{v}_1 \end{array} \right\} - \left\{ \begin{array}{c} \boldsymbol{u}_2 \\ \boldsymbol{v}_2 \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{u}_1 - \boldsymbol{u}_2 \\ \boldsymbol{v}_1 - \boldsymbol{v}_2 \end{array} \right\} \tag{504}$$

The dot and cross products for screws are given by

$$\left\{ \begin{array}{c} \boldsymbol{u}_1 \\ \boldsymbol{v}_1 \end{array} \right\} \cdot \left\{ \begin{array}{c} \boldsymbol{u}_2 \\ \boldsymbol{v}_2 \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{u}_1 \cdot \boldsymbol{s}_2 \\ \boldsymbol{u}_1 \cdot \boldsymbol{v}_2 + \boldsymbol{v}_1 \cdot \boldsymbol{u}_2 \end{array} \right\} \tag{505}$$

$$\left\{ \begin{array}{c} \boldsymbol{u}_1 \\ \boldsymbol{v}_1 \end{array} \right\} \times \left\{ \begin{array}{c} \boldsymbol{u}_2 \\ \boldsymbol{v}_2 \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{u}_1 \times \boldsymbol{u}_2 \\ \boldsymbol{u}_1 \times \boldsymbol{v}_2 + \boldsymbol{v}_1 \times \boldsymbol{u}_2 \end{array} \right\} \tag{506}$$

### 13.5 Screw description of a line

Consider the 3-dimensional Euclidean space $\mathbb{R}^3$ with origin at the point $O$. Consider a line $L$ that passes through the two points $P$ and $Q$ with positions $\vec{p}$ and $\vec{q}$ relative to the origin $O$. The unit vector $\vec{a} = (\vec{q} - \vec{p})/|\vec{q} - \vec{p}|$ is a vector along the line $L$, and is said to be the direction vector of the line. An arbitrary point on the line will have position $\vec{r} = \vec{p} + \lambda\vec{a} = (1 - \lambda)\vec{p} + \lambda\vec{q}$ for some real $\lambda$. The line is therefore determined by the two points $\vec{p}$ and $\vec{q}$. The moment of the line is $\vec{m} = \vec{p} \times \vec{a}$.

The line can be described with the unit direction vector $\vec{a}$ and the moment $\vec{m}$. These two vectors can be represented in terms of a screw as

$$\vec{l} = (\vec{a}, \vec{m}) \tag{507}$$

The moment is perpendicular to the direction vector, which means that $\vec{m} \cdot \vec{a} = 0$. According to convention the description of a line is normalized so that the direction vector is a unit vector satisfying $|\vec{a}| = 1$.

Suppose that the moment is given with respect to the origin of a frame $b$ as $\vec{m}_b = \vec{q}_b \times \vec{a}$. Then the line is given by the vectors $\vec{a}$ and $\vec{m}_b$. Next, suppose that the moment is given with reference to the origion of frame $c$ as $\vec{m}_c = \vec{q}_c \times \vec{a}$ where $\vec{q}_c = \vec{q}_b + \vec{p}_{bc}$ and $\vec{p}_{bc}$ is the position vector from the origin of frame $b$ to the origin of frame $c$. Then the line is given by the vectors $\vec{a}$ and $\vec{m}_c = \vec{m}_b + \vec{p}_{bc} \times \vec{a}$. This shows that tha line is transformed by a screw transformation when the reference point is changed.

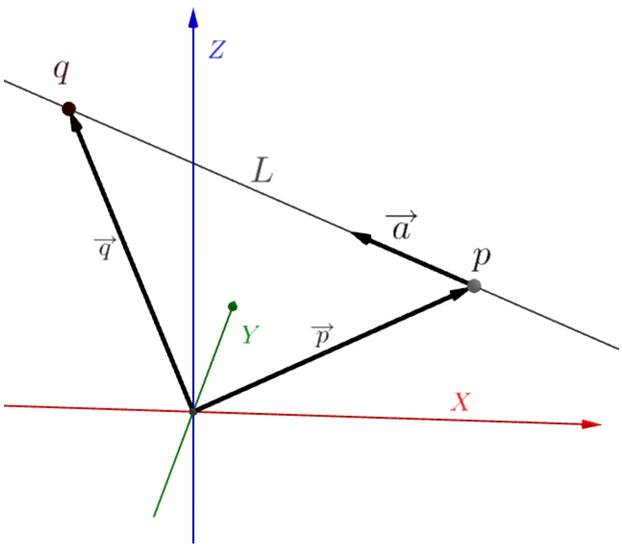



Figure 24: A line $L$ through the points at $\vec{p}$ and $\vec{q}$ with direction vector $\vec{a}$.

### 13.6 Lines and screws

Consider a screw $\vec{s} = (\vec{u}, \vec{v})$. Let the first vector be written $\vec{u} = \alpha\vec{a}$ where $\vec{a}$ is a unit vector and $\alpha = |\vec{u}|$. The second vector can be written

$$\vec{v} = \vec{v}_{||} + \vec{v}_{\perp} \tag{508}$$

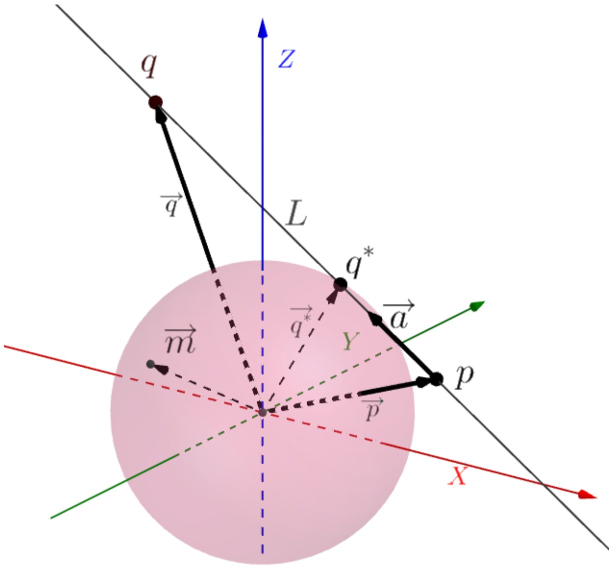



Figure 25: A line $L$ with unit direction vector $\vec{a}$ and moment $\vec{m} = \vec{p} \times \vec{a}$. The radius of the sphere is equal to the magnitude of $\vec{m}$. Note that the line is tangent to this sphere at $\vec{q}^*$, which is the closest point to the origin on the line.

where $\vec{v}_{\|} = (\vec{a} \cdot \vec{v})\vec{a}$ is parallel to the first vector $\vec{u}$, and $\vec{v}_\perp = (\vec{a} \times \vec{v}) \times \vec{a} = -\vec{a} \times (\vec{a} \times \vec{v})$ is perpendicular to $\vec{u}$.

Consider the line $\vec{l} = (\vec{a}, \vec{m})$ where $\vec{m} = \vec{v}_\perp/\alpha$ is the moment of the line. Then $\vec{q} = \vec{a} \times \vec{m} = \vec{a} \times \vec{v}_\perp/\alpha$ is the position of the point on the line that is closest to the origin. Then the second vector can be written

$$\vec{v} = \alpha[\vec{m} + (\vec{a} \cdot \vec{v})\vec{a}] \tag{509}$$

and the screw is

$$\vec{s} = \alpha(\vec{a}, \vec{m} + h\vec{a}) \tag{510}$$

where $h = \vec{a} \cdot \vec{v}/\alpha$ is called the pitch of the screw. This screw is said to have the line $\vec{l} = (\vec{a}, \vec{m})$ as its screw axis. The screw is obtained from its screw axis by multiplication of the line with the scalar $\alpha$, and by adding a term $\alpha h \vec{a}$ to the second vector.

In coordinate form the screw is

$$\mathsf{S} = \left\{ \begin{array}{c} \boldsymbol{u} \\ \boldsymbol{v} \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{u} \\ \boldsymbol{v}_\perp + h\boldsymbol{u} \end{array} \right\} \tag{511}$$

where

$$\boldsymbol{v}_\perp = -\frac{\boldsymbol{u}^\times \boldsymbol{u}^\times}{\boldsymbol{u}^\mathrm{T}\boldsymbol{u}}\boldsymbol{v} \tag{512}$$

is the component of $\boldsymbol{v}$ that is perpendicular to $\boldsymbol{u}$, and

$$h = \frac{\boldsymbol{u}^\mathrm{T}\boldsymbol{v}}{|\boldsymbol{u}|} \tag{513}$$

is the pitch of the screw.

The screw axis is the line

$$\mathsf{L} = \left\{ \begin{array}{c} \boldsymbol{a} \\ \boldsymbol{m} \end{array} \right\} = \frac{1}{\alpha} \left\{ \begin{array}{c} \boldsymbol{u} \\ \boldsymbol{v}_\perp \end{array} \right\}$$

The screw can be expressed in terms of the line parameters as

$$\mathsf{S} = \alpha \left\{ \begin{array}{c} \boldsymbol{a} \\ \boldsymbol{m} + h\boldsymbol{a} \end{array} \right\}$$

It is seen that a screw with zero pitch is a line multiplied with the scalar $\alpha$.

A screw with infinite pitch can be identified with a line at infinity. This is best seen by adjusting the scaling so that $p_s\alpha = 1$. Then

$$\lim_{\substack{p_s \to \infty \\ p_s\alpha = 1}} \left\{ \begin{array}{c} \alpha\boldsymbol{a} \\ \alpha\boldsymbol{q} \times \boldsymbol{a} + \alpha p_s \boldsymbol{a} \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{0} \\ \boldsymbol{a} \end{array} \right\}$$

which is a line at infinity.

## 13.7 Change of reference point of a line by a screw transformation

Consider a line $\vec{l}_c$ with reference point at the origin of a frame $c$. The line is written in the form $\vec{l}_c = (\vec{a}, \vec{m}_c)$ where the moment $\vec{m}_c = \vec{p}_c \times \vec{a}$ depends on the vector $\vec{p}_c$ from the reference point of frame $c$ to the line. The same line can be referenced to the origin of a frame $d$. This the line is represented by the screw $\vec{l}_d = (\vec{a}, \vec{m}_d)$ where the moment is $\vec{m}_d = \vec{p}_d \times \vec{a}$ where $\vec{p}_d$ is the vector from the reference point of $d$ to the line. It follows that

$$\vec{m}_d = \vec{m}_c + \vec{p}_{dc} \times \vec{a} \tag{514}$$

where $\vec{p}_{dc} = \vec{p}_d - \vec{p}_c$ is the vector from $d$ to $c$. The transformation of the line is therefore given by

$$\vec{l}_c = (\vec{a}, \vec{m}_c), \quad \vec{l}_d = (\vec{a}, \vec{m}_c + \vec{p}_{dc} \times \vec{a}) \tag{515}$$

This is a screw transformation.

In the coordinate form the screw transformation will be combined with a coordinate transformation. Consider the line

$$\mathsf{L}_c^c = \left\{ \begin{array}{c} \boldsymbol{a}^c \\ \boldsymbol{m}_c^c \end{array} \right\} \tag{516}$$

The line is referenced to $c$, and it is given in the coordinates of $c$. The same line can be given as

$$\mathsf{L}_d^d = \left\{ \begin{array}{c} \boldsymbol{a}^d \\ \boldsymbol{m}_d^d \end{array} \right\} \tag{517}$$

where the line is referenced to frame $d$ is and is given in the coordinates of $d$. The vectors of the line are transformed according to

$$\boldsymbol{a}^d = \boldsymbol{R}_c^d \boldsymbol{a}^c, \quad \boldsymbol{m}_d^d = \boldsymbol{R}_c^d \boldsymbol{m}_d^d + (\boldsymbol{p}_{dc}^d)^\times \boldsymbol{R}_c^d \boldsymbol{a}^c$$

The screw transformation is therefore given by

$$\mathsf{L}_d^d = \left[ \begin{array}{cc} \boldsymbol{R}_c^d & \boldsymbol{0} \\ (\boldsymbol{p}_{dc}^d)^\times \boldsymbol{R}_c^d & \boldsymbol{R}_c^d \end{array} \right] \mathsf{L}_c^c \tag{518}$$

Note that in the coordinate form the screw transformation is a coordinate transformation due to $\boldsymbol{R}_c^d$ and a transformation of the reference point, which is due to the term $(\boldsymbol{p}_{dc}^d)^\times$.

Note that the screw transformation can have a coordinate transformation that is different from the transformation of reference point. As an example from robot kinematics, the line of joint axis $i$ is given by $\mathsf{L}_i^i$ when it is referenced to frame $i$ and given in the coordinates of $i$. This line is transformed by the screw transformation

$$\mathsf{L}_e^0 = \begin{bmatrix} \boldsymbol{R}_i^0 & \mathbf{0} \\ (\boldsymbol{p}_{cd}^0)^\times \boldsymbol{R}_i^0 & \boldsymbol{R}_i^0 \end{bmatrix} \mathsf{L}_i^i \tag{519}$$

to be referenced to the end effector frame $e$ and given in the coordinates of of the base frame 0.

### 13.8 Dual numbers

Screws are often represented i dual form, which gives efficient rules for computations. To introduce this it is necessary to first give a basic introduction to dual numbers, starting with dual scalars. Dual scalars and dual functions are useful in several other applications, like automatic differentiation of functions.

A dual number $\hat{a}$ is given by

$$\hat{a} = a + \varepsilon a'$$

where $a$ and $a'$ are real numbers and $\varepsilon$ is the dual unit that satisfies $\varepsilon \neq 0$ and $\varepsilon^2 = 0$.

Let two dual numbers be given by $\hat{a} = a + \varepsilon a'$ and $\hat{b} = b + \varepsilon b'$. Then addition and subtraction is done component-wise

$$\hat{a} + \hat{b} = a + b + \varepsilon(a' + b') \tag{520}$$

$$\hat{a} - \hat{b} = a - b + \varepsilon(a' - b') \tag{521}$$

Multiplication is performed as if the numbers were polynomials in $\varepsilon$, which gives

$$\hat{a}\hat{b} = (a + \varepsilon a')(b + \varepsilon b') = ab + \varepsilon(ab' + a'b) \tag{522}$$

The square of a dual number is found by multiplication to be

$$\hat{a}^2 = \hat{a}\hat{a} = (a + \varepsilon a')(a + \varepsilon a') = a^2 + 2\varepsilon aa' \tag{523}$$

Division is done as follows:

$$\frac{\hat{a}}{\hat{b}} = \frac{a + \varepsilon a'}{b + \varepsilon b'} \tag{524}$$

$$= \frac{a + \varepsilon a'}{b + \varepsilon b'} \frac{b - \varepsilon b'}{b - \varepsilon b'} \tag{525}$$

$$= \frac{ab + \varepsilon(a'b - ab')}{b^2} \tag{526}$$

$$= \frac{a}{b} + \varepsilon \frac{a'b - ab'}{b^2} \tag{527}$$

The inverse of a dual number is therefore

$$\frac{1}{\hat{x}} = \frac{1}{x + \varepsilon x'} = \frac{1}{x} - \varepsilon \frac{x'}{x^2} \tag{528}$$

The square root of a dual number $\hat{a} = a + \varepsilon a'$ with $a \geq 0$ is

$$\sqrt{\hat{a}} = \sqrt{a} + \varepsilon \frac{a'}{2\sqrt{a}} \tag{529}$$

which is verified by

$$\left( \sqrt{a} + \varepsilon \frac{a'}{2\sqrt{a}} \right)^2 = a + \varepsilon 2\sqrt{a} \frac{a'}{2\sqrt{a}} = a + \varepsilon a' = \hat{a} \tag{530}$$

This is consistent with the expression of square of a dual number $\hat{a} = a + \varepsilon a'$, which is seen from

$$\sqrt{\hat{a}^2} = \sqrt{a^2 + 2\varepsilon a a'} = \sqrt{a^2} + \varepsilon \frac{2aa'}{2\sqrt{a^2}} = a + \varepsilon a' = \hat{a} \tag{531}$$

### 13.9 Screws represented by dual numbers

A screw

$$\mathsf{S} = \left\{ \begin{array}{c} \boldsymbol{u} \\ \boldsymbol{v} \end{array} \right\}$$

can be represented by the dual vector

$$\hat{\boldsymbol{s}} = \boldsymbol{u} + \varepsilon \boldsymbol{v} = \boldsymbol{s} + \varepsilon \boldsymbol{s}'$$

where $\boldsymbol{s} = \boldsymbol{u}$ and $\boldsymbol{s}' = \boldsymbol{v}$. Computations can be performed on the dual vectors in the same way as computations are done on real vectors with the additional rule that $\varepsilon^2 = 0$. This is obviously the case for addition and subtraction of the two screws $\hat{\boldsymbol{s}}_1 = \boldsymbol{s}_1 + \varepsilon \boldsymbol{s}'_1$ $\hat{\boldsymbol{s}}_2 = \boldsymbol{s}_2 + \varepsilon \boldsymbol{s}'_2$, which gives

$$\hat{\boldsymbol{s}}_1 + \hat{\boldsymbol{s}}_2 = \boldsymbol{s}_1 + \boldsymbol{s}_2 + \varepsilon(\boldsymbol{s}'_1 + \boldsymbol{s}'_2) \tag{532}$$

$$\hat{\boldsymbol{s}}_1 - \hat{\boldsymbol{s}}_2 = \boldsymbol{s}_1 - \boldsymbol{s}_2 + \varepsilon(\boldsymbol{s}'_1 - \boldsymbol{s}'_2) \tag{533}$$

while the dot product is

$$\hat{\boldsymbol{s}}_1 \cdot \hat{\boldsymbol{s}}_2 = (\boldsymbol{s}_1 + \varepsilon \boldsymbol{s}'_1) \cdot (\boldsymbol{s}_2 + \varepsilon \boldsymbol{s}'_2) \tag{534}$$

$$= \boldsymbol{s}_1 \cdot \boldsymbol{s}_2 + \varepsilon(\boldsymbol{s}_1 \cdot \boldsymbol{s}'_2 + \boldsymbol{s}'_1 \cdot \boldsymbol{s}_2) + \varepsilon^2 \boldsymbol{s}'_1 \cdot \boldsymbol{s}'_2 \tag{535}$$

$$= \boldsymbol{s}_1 \cdot \boldsymbol{s}_2 + \varepsilon(\boldsymbol{s}_1 \cdot \boldsymbol{s}'_2 + \boldsymbol{s}'_1 \cdot \boldsymbol{s}_2) \tag{536}$$

Likewise, the cross product is found to be

$$\hat{\boldsymbol{s}}_1 \times \hat{\boldsymbol{s}}_2 = (\boldsymbol{s}_1 + \varepsilon \boldsymbol{s}'_1) \times (\boldsymbol{s}_2 + \varepsilon \boldsymbol{s}'_2) \tag{537}$$

$$= \boldsymbol{s}_1 \times \boldsymbol{s}_2 + \varepsilon(\boldsymbol{s}_1 \times \boldsymbol{s}'_2 + \boldsymbol{s}'_1 \times \boldsymbol{s}_2) + \varepsilon^2 \boldsymbol{s}'_1 \times \boldsymbol{s}'_2 \tag{538}$$

$$= \boldsymbol{s}_1 \times \boldsymbol{s}_2 + \varepsilon(\boldsymbol{s}_1 \times \boldsymbol{s}'_2 + \boldsymbol{s}'_1 \times \boldsymbol{s}_2) \tag{539}$$

### 13.10 Dual representations of a line

The screw vector representation of a line $\vec{l} = (\vec{a}, \vec{m})$ is given by

$$\mathsf{L} = \left\{ \begin{array}{c} \boldsymbol{a} \\ \boldsymbol{m} \end{array} \right\} \tag{540}$$

where $\boldsymbol{a}$ is the unit direction vector of the line and $\boldsymbol{m} = \boldsymbol{p}^{\times}\boldsymbol{a}$ is the moment, where $\boldsymbol{p}$ is a point on the line.

The dual form of the line is

$$\hat{\boldsymbol{l}} = \boldsymbol{a} + \varepsilon \boldsymbol{m} \tag{541}$$

### 13.11 Dual functions

A function $f(x)$ of $x$ can be defined as a function $f(\hat{x})$ of the dual number $\hat{x} = x + \varepsilon x'$ through the Taylor series in $x'$ about $x' = 0$. This gives

$$f(\hat{x}) = f(x + \varepsilon x') = f(x) + \varepsilon \frac{df(x)}{dx} x' + \frac{1}{2!}\varepsilon^2 \frac{d^2 f(x)}{dx^2}(x')^2 + \dots$$

Then because $\varepsilon^2 = 0$, this gives the remarkable result

$$f(\hat{x}) = f(x) + \varepsilon \frac{df(x)}{dx} x'$$

Note that this is not an approximation, the function of the dual number is actually the function of the real part plus a dual part which is the derivative of the function.

Dual functions are important in screw theory, and is in particular used for the dual angle

$$\hat{\theta} = \theta + \varepsilon d$$

which is an angle $\theta$ plus a dual part which is a distance $d$. The sine and the cosine of the dual angle is found from the Taylor series expansion to be

$$\sin\hat{\theta} = \sin\theta + \varepsilon d \cos\theta \tag{542}$$

$$\cos\hat{\theta} = \cos\theta - \varepsilon d \sin\theta \tag{543}$$

$$\tan\hat{\theta} = \tan\theta + \varepsilon \frac{d}{\cos^2\theta} \tag{544}$$

This is in agreement with

$$\tan\hat{\theta} = \frac{\sin\hat{\theta}}{\cos\hat{\theta}} = \frac{\sin\theta}{\cos\theta} + \varepsilon \frac{d\sin^2\theta + d\cos^2\theta}{\cos^2\theta} = \tan\theta + \varepsilon \frac{d}{\cos^2\theta} \tag{545}$$

It is noted that

$$\sin^2\hat{\theta} + \cos^2\hat{\theta} = 1 \tag{546}$$

which is verified by the calculation

$$\sin^2\hat\theta+\cos^2\hat\theta = (\sin\theta + d\varepsilon\cos\theta)^2 + (\cos\theta - d\varepsilon\sin\theta)^2 \tag{547}$$

$$= \sin^2\theta + 2d\varepsilon\sin\theta\cos\theta + \cos^2\theta - 2d\varepsilon\sin\theta\cos\theta \tag{548}$$

$$= \sin^2\theta + \cos^2\theta \tag{549}$$

Moreover, there are dual trigonometric identities in the form

$$\sin\hat\theta = 2\sin\frac{\hat\theta}{2}\cos\frac{\hat\theta}{2} \tag{550}$$

$$\cos\hat\theta = \cos^2\frac{\hat\theta}{2} - \sin^2\frac{\hat\theta}{2} \tag{551}$$

which are direct extensions of the usual trigonometric identities. This is validated by

$$2\sin\frac{\hat\theta}{2}\cos\frac{\hat\theta}{2} = 2\left(\sin\frac{\theta}{2} + \varepsilon\frac{d}{2}\cos\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} - \varepsilon\frac{d}{2}\sin\frac{\theta}{2}\right) = \sin\theta + \varepsilon d\cos\theta \tag{552}$$

and

$$\cos^2\frac{\hat\theta}{2} - \sin^2\frac{\hat\theta}{2} = (\cos\theta - d\varepsilon\sin\theta)^2 - (\sin\theta + d\varepsilon\cos\theta)^2 \tag{553}$$

$$= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} - 2d\varepsilon\sin\frac{\theta}{2}\cos\frac{\theta}{2} \tag{554}$$

$$= \cos\theta - \varepsilon d\sin\theta \tag{555}$$

It is noted that

$$2\sin\hat\theta\cos\hat\theta = \sin 2\hat\theta = \sin 2\theta + \varepsilon d\cos 2\theta \tag{556}$$

□

### 13.12 The scalar product and the vector product of two lines

Consider two lines $\hat{\boldsymbol{l}}_1 = \boldsymbol{a}_1 + \varepsilon\boldsymbol{q}_1\times\boldsymbol{a}_1$ and $\hat{\boldsymbol{l}}_2 = \boldsymbol{a}_2 + \varepsilon\boldsymbol{q}_2\times\boldsymbol{a}_2$. The cross product of the two lines is given by

$$\hat{\boldsymbol{l}}_1\times\hat{\boldsymbol{l}}_2 = (\boldsymbol{a}_1 + \varepsilon\boldsymbol{q}_1\times\boldsymbol{a}_1)\times(\boldsymbol{a}_2 + \varepsilon\boldsymbol{q}_2\times\boldsymbol{a}_2) \tag{557}$$

$$= \boldsymbol{a}_1\times\boldsymbol{a}_2 + \varepsilon[\boldsymbol{a}_1\times(\boldsymbol{q}_2\times\boldsymbol{a}_2) - \boldsymbol{a}_2\times(\boldsymbol{q}_1\times\boldsymbol{a}_1)] \tag{558}$$

Let $\boldsymbol{n}$ be the unit vector that is normal to the direction vectors $\boldsymbol{a}_2$ and $\boldsymbol{a}_2$ so that $\boldsymbol{a}_1\times\boldsymbol{a}_2 = \boldsymbol{n}\sin\theta$, and let $\hat{\boldsymbol{n}} = \boldsymbol{n} + \varepsilon\boldsymbol{q}_1\times\boldsymbol{n}$ be the line with direction vector $\boldsymbol{n}$ through the point $\boldsymbol{q}_1$ on $\hat{\boldsymbol{l}}_1$. Further, suppose that the two points $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are selected so that $\boldsymbol{q}_2 = \boldsymbol{q}_1 + d\boldsymbol{n}$ where $d$ is the distance between the lines. Then

$$\hat{\boldsymbol{l}}_1\times\hat{\boldsymbol{l}}_2 = \boldsymbol{n}\sin\theta + \varepsilon[\boldsymbol{a}_1\times((\boldsymbol{q}_1 + d\boldsymbol{n})\times\boldsymbol{a}_2) - \boldsymbol{a}_2\times(\boldsymbol{q}_1\times\boldsymbol{a}_1)] \tag{559}$$

$$= \boldsymbol{n}\sin\theta + \varepsilon[\boldsymbol{a}_1\times(\boldsymbol{q}_1\times\boldsymbol{a}_2) + \boldsymbol{a}_2\times(\boldsymbol{a}_1\times\boldsymbol{q}_1) + d\boldsymbol{a}_1\times(\boldsymbol{n}\times\boldsymbol{a}_2)] \tag{560}$$

$$= \boldsymbol{n}\sin\theta + \varepsilon[\boldsymbol{q}_1\times(\boldsymbol{a}_1\times\boldsymbol{a}_2) + d\boldsymbol{a}_1\times(\boldsymbol{n}\times\boldsymbol{a}_2)] \tag{561}$$

$$= \boldsymbol{n}\sin\theta + \varepsilon(\sin\theta\boldsymbol{q}_1\times\boldsymbol{n} + d\cos\theta\boldsymbol{n}) \tag{562}$$

$$= (\sin\theta + \varepsilon d\cos\theta)(\boldsymbol{n} + \varepsilon\boldsymbol{q}_1\times\boldsymbol{n}) \tag{563}$$

$$= \sin\hat\theta\hat{\boldsymbol{n}} \tag{564}$$

In the derivation the Jacobi identity $\boldsymbol{a}_1 \times (\boldsymbol{q}_1 \times \boldsymbol{a}_2) + \boldsymbol{a}_2 \times (\boldsymbol{a}_1 \times \boldsymbol{q}_1) + \boldsymbol{q}_1 \times (\boldsymbol{a}_2 \times \boldsymbol{a}_1) = 0$ is used, and the identity $\boldsymbol{a}_1 \times (\boldsymbol{n} \times \boldsymbol{a}_2) = (\boldsymbol{a}_1 \cdot \boldsymbol{a}_2)\boldsymbol{n} - (\boldsymbol{a}_1 \cdot \boldsymbol{n})\boldsymbol{a}_2 = \cos\theta \boldsymbol{n}$ of the triple vector product.

The scalar product of the two lines is given by

$$\begin{aligned} \hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2 =& (\boldsymbol{a}_1 + \varepsilon \boldsymbol{q}_1 \times \boldsymbol{a}_1) \cdot (\boldsymbol{a}_2 + \varepsilon \boldsymbol{q}_2 \times \boldsymbol{a}_2) && (565) \\ =& \boldsymbol{a}_1 \cdot \boldsymbol{a}_2 + \varepsilon[\boldsymbol{a}_1 \cdot (\boldsymbol{q}_2 \times \boldsymbol{a}_2) + \boldsymbol{a}_2 \cdot (\boldsymbol{q}_1 \times \boldsymbol{a}_1)] && (566) \end{aligned}$$

The triple scalar product rule $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = (\boldsymbol{w} \times \boldsymbol{u}) \cdot \boldsymbol{v} = (\boldsymbol{v} \times \boldsymbol{w}) \cdot \boldsymbol{u}$ can be used to give

$$\begin{aligned} \hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2 =& \boldsymbol{a}_1 \cdot \boldsymbol{a}_2 + \varepsilon[\boldsymbol{q}_2 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_1) + \boldsymbol{q}_1 \cdot (\boldsymbol{a}_1 \times \boldsymbol{a}_2)] && (567) \\ =& \boldsymbol{a}_1 \cdot \boldsymbol{a}_2 - \varepsilon[(\boldsymbol{q}_2 - \boldsymbol{q}_1) \cdot (\boldsymbol{a}_1 \times \boldsymbol{a}_2)] && (568) \\ =& \boldsymbol{a}_1 \cdot \boldsymbol{a}_2 - \varepsilon[d\boldsymbol{n} \cdot (\boldsymbol{n} \sin\theta)] && (569) \\ =& \cos\theta - \varepsilon d \sin\theta && (570) \\ =& \cos\hat{\theta} && (571) \end{aligned}$$

This shows that the vector product of two lines $\hat{\boldsymbol{l}}_1$ and $\hat{\boldsymbol{l}}_2$ is given by

$$\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2 = \sin\hat{\theta}\hat{\boldsymbol{n}} \quad (572)$$

where $\hat{\boldsymbol{n}}$ is the common normal of the two lines. The scalar product is given by

$$\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2 = \cos\hat{\theta} \quad (573)$$

The dual angle $\hat{\theta} = \theta + \varepsilon d$ is given by the rotation angle between the two lines around the common normal $\hat{\boldsymbol{n}}$, and the translation $d$ from $\hat{\boldsymbol{l}}_1$ to $\hat{\boldsymbol{l}}_2$ along the common normal.

### 13.13 Calculation of the common normal

Consider two lines $\hat{\boldsymbol{l}}_1 = \boldsymbol{l}_1 + \varepsilon \boldsymbol{l'}_1$ and $\hat{\boldsymbol{l}}_2 = \boldsymbol{l}_2 + \varepsilon \boldsymbol{l'}_2$. The cross product is

$$\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2 = \boldsymbol{l}_1 \times \boldsymbol{l}_2 + \varepsilon(\boldsymbol{l}_1 \times \boldsymbol{l'}_2 + \boldsymbol{l'}_1 \times \boldsymbol{l}_2) \quad (574)$$

The cross product was found to be

$$\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2 = \sin\hat{\theta}\boldsymbol{l}_n \quad (575)$$

where $\boldsymbol{l}_n$ is the common normal of the two lines, and the displacement from $\hat{\boldsymbol{l}}_1$ to $\boldsymbol{l}_2$ is given by the dual angle $\hat{\theta} = \theta + \varepsilon d$ about the common normal. Let $\hat{s} = \sin\hat{\theta}\boldsymbol{l}_n$ be the screw of the cross product. Then

$$\hat{s} \cdot \hat{s} = (\sin\hat{\theta}\boldsymbol{l}_n) \cdot (\sin\hat{\theta}\boldsymbol{l}_n) = \sin\hat{\theta}^2(\boldsymbol{l}_n \cdot \boldsymbol{l}_n) = \sin\hat{\theta}^2 \quad (576)$$

and it follows that

$$\sqrt{\hat{s} \cdot \hat{s}} = \sin\hat{\theta} \quad (577)$$

and the common normal is found from

$$\boldsymbol{l}_n = \frac{\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2}{\sqrt{(\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2) \cdot (\hat{\boldsymbol{l}}_1 \times \hat{\boldsymbol{l}}_2)}} \quad (578)$$

### 13.14 The intersection of two lines

Consider two lines $\hat{\boldsymbol{l}}_1 = \boldsymbol{l}_1 + \varepsilon \boldsymbol{l}_1'$ and $\hat{\boldsymbol{l}}_2 = \boldsymbol{l}_2 + \varepsilon \boldsymbol{l}_2'$ with scalar product

$$\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2 = \cos\hat{\theta} \tag{579}$$

It is seen that if

$$\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2 = 0 \tag{580}$$

then $\cos\hat{\theta} = \cos\theta - \varepsilon d \sin\theta = 0$ which implies that $\cos\theta = 0$ and $d = 0$, which implies that the two lines are perpendicular, and in addition, that they intersect. Note that the point of intersection is not determined by the condition (579).

Next, consider two screws $\hat{\boldsymbol{s}}_1 = \hat{\alpha}_1 \hat{\boldsymbol{l}}_1$ and $\hat{\boldsymbol{s}}_2 = \hat{\alpha}_2 \hat{\boldsymbol{l}}_2$ with intersecting screw axes in form of the lines $\hat{\boldsymbol{l}}_1$ and $\hat{\boldsymbol{l}}_2$. Then the scalar product of the screws is zero, which is verified with the calculation

$$\hat{\boldsymbol{s}}_1 \cdot \hat{\boldsymbol{s}}_2 = (\hat{\alpha}_1 \hat{\boldsymbol{l}}_1) \cdot (\hat{\alpha}_2 \hat{\boldsymbol{l}}_2) = \hat{\alpha}_1 \hat{\alpha}_2 (\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2) = 0 \tag{581}$$

which again implies that $\cos\hat{\theta} = 0$. This means that the two screws intersects at a right angle.

The two lines $\hat{\boldsymbol{l}}_1$ and $\hat{\boldsymbol{l}}_2$ are said to be reciprocal if the scalar product is real, that is, if $\hat{\boldsymbol{l}}_1 \cdot \hat{\boldsymbol{l}}_2 = \cos\theta$ and

$$d \sin\theta = 0 \tag{582}$$

Two cases are possible. The lines are reciprocal if $d = 0$ and the two lines intersect, or if $\sin\theta = 0$ and the two lines are parallel.

Consider the two screws $\hat{\boldsymbol{s}}_1$ and $\hat{\boldsymbol{s}}_2$, and let $\hat{\alpha}_i = \alpha_i(1 + \varepsilon p_i)$. Then

$$\hat{\boldsymbol{s}}_1 \cdot \hat{\boldsymbol{s}}_2 = \alpha_1 \alpha_2 [\cos\theta + \varepsilon((p_1 + p_2)\cos\theta - d \sin\theta)] = 0 \tag{583}$$

The two screws are reciprocal if

$$(p_1 + p_2)\cos\theta - d \sin\theta = 0 \tag{584}$$

which implies that

$$d \tan\theta = p_1 + p_2 \tag{585}$$

### 13.15 Dual magnitude of a screw

Let the line $\mathsf{L}$ be given by dual vectors as

$$\hat{\boldsymbol{l}} = \boldsymbol{a} + \varepsilon \boldsymbol{q} \times \boldsymbol{a} \tag{586}$$

where $\boldsymbol{a}$ is a unit vector and the magnitude of the line is unity. It is seen that

$$\hat{\boldsymbol{l}} \cdot \hat{\boldsymbol{l}} = \boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2 = 1 \tag{587}$$

The magnitude of a line is therefore unity.

Let the screw $\mathsf{S}$ be given in the dual formulation by

$$\hat{\boldsymbol{s}} = \boldsymbol{s} + \varepsilon \boldsymbol{s}' = \alpha[\boldsymbol{a} + \varepsilon(\boldsymbol{q} \times \boldsymbol{a} + p_s \boldsymbol{a})] \tag{588}$$

where $\boldsymbol{s} = \alpha \boldsymbol{a}$ and $\boldsymbol{s}' = \alpha(\boldsymbol{q} \times \boldsymbol{a} + p_s \boldsymbol{a})$. Then the screw can be written

$$\hat{\boldsymbol{s}} = \hat{\alpha} \hat{\boldsymbol{l}} \tag{589}$$

where

$$\hat{\alpha} = \alpha + \varepsilon \alpha p_s \tag{590}$$

is the dual magnitude of the screw $\hat{\boldsymbol{s}}$. This is seen from the calculation

$$\hat{\alpha} \hat{\boldsymbol{l}} = (\alpha + \varepsilon \alpha p_s)(\boldsymbol{a} + \varepsilon \boldsymbol{q} \times \boldsymbol{a}) = \alpha \boldsymbol{a} + \varepsilon(\alpha \boldsymbol{q} \times \boldsymbol{a} + \alpha p_s \boldsymbol{a}) = \hat{\boldsymbol{s}} \tag{591}$$

The dual magnitude is found from

$$\hat{\alpha} = \sqrt{\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{s}}} \tag{592}$$

which follows from $\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{s}} = \hat{\alpha}^2 \hat{\boldsymbol{l}} \cdot \hat{\boldsymbol{l}} = \hat{\alpha}^2$. This is verified with the following computation

$$\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{s}} = \alpha[\boldsymbol{a} + \varepsilon(\boldsymbol{q} \times \boldsymbol{a} + p_s \boldsymbol{a})] \cdot \alpha[\boldsymbol{a} + \varepsilon(\boldsymbol{q} \times \boldsymbol{a} + p_s \boldsymbol{a})] = \alpha^2(\boldsymbol{a} \cdot \boldsymbol{a} + 2\varepsilon p_s \boldsymbol{a} \cdot \boldsymbol{a}) = \alpha^2 + 2\varepsilon \alpha^2 p_s \tag{593}$$

which gives

$$\sqrt{\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{s}}} = \sqrt{\alpha^2 + 2\varepsilon \alpha^2 p_s} = \alpha + \varepsilon \alpha p_s = \hat{\alpha} \tag{594}$$

If the screw $\hat{\boldsymbol{s}}$ is given, then the line $\hat{\boldsymbol{l}}$ can be found from

$$\hat{\boldsymbol{l}} = \frac{\hat{\boldsymbol{s}}}{\sqrt{\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{s}}}} \tag{595}$$

This is verified by

$$\frac{\hat{\boldsymbol{s}}}{\sqrt{\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{s}}}} = \frac{\hat{\boldsymbol{s}}}{\hat{\alpha}} = \frac{\boldsymbol{s} + \varepsilon \boldsymbol{s}'}{\alpha + \varepsilon \alpha p_s} = \frac{\boldsymbol{s}}{\alpha} + \varepsilon \frac{\alpha \boldsymbol{s}' - \alpha p_s \boldsymbol{s}}{\alpha^2} = \boldsymbol{l} + \varepsilon \boldsymbol{q} \times \boldsymbol{a} = \hat{\boldsymbol{l}} \tag{596}$$

### 13.16 Screw description of a coordinate frame

Consider a coordinate frame with orthogonal unit axes $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ and let the position of the origin be $\boldsymbol{o}$. Then the coordinate axes will be the lines

$$\hat{\boldsymbol{e}}_1 = \boldsymbol{e}_1 + \varepsilon \boldsymbol{e}'_1 \tag{597}$$

$$\hat{\boldsymbol{e}}_2 = \boldsymbol{e}_2 + \varepsilon \boldsymbol{e}'_2 \tag{598}$$

$$\hat{\boldsymbol{e}}_3 = \boldsymbol{e}_3 + \varepsilon \boldsymbol{e}'_3 \tag{599}$$

where $\boldsymbol{e}'_i = \boldsymbol{o} \times \boldsymbol{e}_i$ is the moment of the coordinate line. Note that with this screw discretion of the coordinate axes, the position of the origin of the frame is encoded in the representation through the moment.

It is noted that the common normal of the coordinate lines $\hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_2$ is $\hat{\boldsymbol{e}}_3$. In the same way, the common normal of $\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_3$ is $\hat{\boldsymbol{e}}_1$, while the common normal of $\hat{\boldsymbol{e}}_3$ and $\hat{\boldsymbol{e}}_1$ is $\hat{\boldsymbol{e}}_2$. From the general expression for the cross product of lines, it follows that

$$\hat{\boldsymbol{e}}_1 \times \hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{e}}_3, \quad \hat{\boldsymbol{e}}_2 \times \hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{e}}_1 \quad \text{and} \quad \hat{\boldsymbol{e}}_3 \times \hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{e}}_2 \tag{600}$$

### 13.17 Decomposition of a screw along the coordinate axes

Consider the screw

$$\hat{\boldsymbol{s}} = \boldsymbol{s} + \varepsilon \boldsymbol{s}' \tag{601}$$

The dual coordinate of $\hat{\boldsymbol{s}}$ along the coordinate axis $\hat{\boldsymbol{e}}_i$ is

$$\hat{s}_i = \hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{e}}_i = \boldsymbol{s} \cdot \boldsymbol{e}_i + \varepsilon(\boldsymbol{s} \cdot \boldsymbol{e}'_i + \boldsymbol{s}' \cdot \boldsymbol{e}_i) \tag{602}$$

Consider

$$\hat{s}_i \hat{\boldsymbol{e}}_i = [\boldsymbol{s} \cdot \boldsymbol{e}_i + \varepsilon(\boldsymbol{s} \cdot \boldsymbol{e}'_i + \boldsymbol{s}' \cdot \boldsymbol{e}_i)](\boldsymbol{e}_i + \varepsilon \boldsymbol{e}'_i) \tag{603}$$

$$= (\boldsymbol{s} \cdot \boldsymbol{e}_i)\boldsymbol{e}_i + \varepsilon[(\boldsymbol{s} \cdot \boldsymbol{e}'_i)\boldsymbol{e}_i + (\boldsymbol{s}' \cdot \boldsymbol{e}_i))\boldsymbol{e}_i + (\boldsymbol{s} \cdot \boldsymbol{e}_i)\boldsymbol{e}'_i] \tag{604}$$

An interesting result is found by using the two results

$$(\boldsymbol{s} \cdot \boldsymbol{e}'_i)\boldsymbol{e}_i = (\boldsymbol{s} \cdot (\boldsymbol{o} \times \boldsymbol{e}_i))\boldsymbol{e}_i = (\boldsymbol{e}_i \cdot (\boldsymbol{s} \times \boldsymbol{o}))\boldsymbol{e}_i \tag{605}$$

$$(\boldsymbol{s} \cdot \boldsymbol{e}_i)\boldsymbol{e}'_i = (\boldsymbol{s} \cdot \boldsymbol{e}_i)\boldsymbol{o} \times \boldsymbol{e}_i = -(\boldsymbol{s} \cdot \boldsymbol{e}_i)\boldsymbol{e}_i \times \boldsymbol{o} \tag{606}$$

Summation over the three dimensions gives the two results

$$\sum_{i=0}^{3}(\boldsymbol{s} \cdot \boldsymbol{e}'_i)\boldsymbol{e}_i = \sum_{i=0}^{3}(\boldsymbol{e}_i \cdot (\boldsymbol{s} \times \boldsymbol{o}))\boldsymbol{e}_i = \boldsymbol{s} \times \boldsymbol{o} \tag{607}$$

$$\sum_{i=0}^{3}(\boldsymbol{s} \cdot \boldsymbol{e}_i)\boldsymbol{e}'_i = -\sum_{i=0}^{3}(\boldsymbol{s} \cdot \boldsymbol{e}_i)\boldsymbol{e}_i \times \boldsymbol{o} = -\boldsymbol{s} \times \boldsymbol{o} \tag{608}$$

Then, it follows that

$$\sum_{i=0}^{3} \hat{s}_i \hat{\boldsymbol{e}}_i = \sum_{i=0}^{3}(\boldsymbol{s} \cdot \boldsymbol{e}_i)\boldsymbol{e}_i + \varepsilon \sum_{i=0}^{3}(\boldsymbol{s}' \cdot \boldsymbol{e}_i)\boldsymbol{e}_i = \boldsymbol{s} + \varepsilon \boldsymbol{s}' \tag{609}$$

This shows that

$$\hat{\boldsymbol{s}} = \hat{s}_1 \hat{\boldsymbol{e}}_1 + \hat{s}_2 \hat{\boldsymbol{e}}_2 + \hat{s}_3 \hat{\boldsymbol{e}}_3 \tag{610}$$

This means that a screw can be decomposed in its components along the coordinate axes $\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3$ where the dual component along the coordinates axis $\hat{\boldsymbol{e}}_i$ is $\hat{s}_i = \hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{e}}_i$.

### 13.18 Decomposition of a line along the coordinate axes

Consider the line

$$\hat{\boldsymbol{l}} = \boldsymbol{l} + \varepsilon \boldsymbol{l}' \tag{611}$$

The line is described in terms of the coordinate axes $\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3$ as

$$\hat{\boldsymbol{l}} = \hat{l}_1 \hat{\boldsymbol{e}}_1 + \hat{l}_2 \hat{\boldsymbol{e}}_2 + \hat{l}_3 \hat{\boldsymbol{e}}_3 \tag{612}$$

where the coordinates are

$$\hat{l}_i = \hat{\boldsymbol{l}} \cdot \hat{\boldsymbol{e}}_i \tag{613}$$

**Example:**

Suppose that the line is displaced from $\hat{\boldsymbol{e}}_1$ by a dual angle $\hat{\theta} = \theta + \varepsilon d$ about $\hat{\boldsymbol{e}}_3$, where $\hat{\boldsymbol{e}}_3$ is the common normal of $\hat{\boldsymbol{l}}$ and $\hat{\boldsymbol{e}}_1$. Then

$$\hat{\boldsymbol{l}} = (\hat{\boldsymbol{l}} \cdot \hat{\boldsymbol{e}}_1)\hat{\boldsymbol{e}}_1 + (\hat{\boldsymbol{l}} \cdot \hat{\boldsymbol{e}}_2)\hat{\boldsymbol{e}}_2 \tag{614}$$

Note that there is no component along $\hat{\boldsymbol{e}}_3$, as this coordinate axis is normal to $\hat{\boldsymbol{l}}$. Then

$$\hat{\boldsymbol{l}} \cdot \hat{\boldsymbol{e}}_1 = \cos\hat{\theta}, \quad \hat{\boldsymbol{l}} \cdot \hat{\boldsymbol{e}}_2 = \sin\hat{\theta} \tag{615}$$

It follows that

$$\hat{\boldsymbol{l}} = \cos\hat{\theta}\hat{\boldsymbol{e}}_1 + \sin\hat{\theta}\hat{\boldsymbol{e}}_2 \tag{616}$$

□

**Example:**

A direct computation of the same result is done as follows: Suppose that the line $\hat{\boldsymbol{l}}$ is is initially aligned with $\hat{\boldsymbol{e}}_1$, and the displaced by a dual angle $\hat{\theta} = \theta + \varepsilon d$ about $\hat{\boldsymbol{e}}_3$. This means that

$$\hat{\boldsymbol{l}} = \boldsymbol{s} + \varepsilon(\boldsymbol{o} + d\boldsymbol{e}_3) \times \boldsymbol{s} \tag{617}$$

where $\boldsymbol{s} = \cos\theta\boldsymbol{e}_1 + \sin\theta\boldsymbol{e}_2$ is the resulting vector when $\boldsymbol{e}_1$ is rotated by an angle $\theta$ around the $z$ axis. This gives

$$\begin{aligned}
\hat{\boldsymbol{l}} =& \cos\theta\boldsymbol{e}_1 + \sin\theta\boldsymbol{e}_2 + \varepsilon(\boldsymbol{o} + d\boldsymbol{e}_3) \times (\cos\theta\boldsymbol{e}_1 + \sin\theta\boldsymbol{e}_2) && (618)\\
=& \cos\theta\boldsymbol{e}_1 + \sin\theta\boldsymbol{e}_2 + \varepsilon(\cos\theta\boldsymbol{o} \times \boldsymbol{e}_1 + \sin\theta\boldsymbol{o} \times \boldsymbol{e}_2 + d\cos\theta\boldsymbol{e}_2 - d\sin\theta\boldsymbol{e}_1) && (619)\\
=& (\cos\theta - \varepsilon d\sin\theta)\boldsymbol{e}_1 + (\sin\theta + \varepsilon d\cos\theta)\boldsymbol{e}_2 + \varepsilon(\cos\theta\boldsymbol{o} \times \boldsymbol{e}_1 + \sin\theta\boldsymbol{o} \times \boldsymbol{e}_2) && (620)\\
=& \cos\hat{\theta}\boldsymbol{e}_1 + \varepsilon\cos\theta\boldsymbol{o} \times \boldsymbol{e}_1 + \sin\hat{\theta}\boldsymbol{e}_2 + \varepsilon\sin\theta\boldsymbol{o} \times \boldsymbol{e}_2 && (621)\\
=& \cos\hat{\theta}(\boldsymbol{e}_1 + \boldsymbol{o} \times \boldsymbol{e}_1) + \sin\hat{\theta}(\boldsymbol{e}_2 + \boldsymbol{o} \times \boldsymbol{e}_2) && (622)\\
=& \cos\hat{\theta}\hat{\boldsymbol{e}}_1 + \sin\hat{\theta}\hat{\boldsymbol{e}}_2 && (623)
\end{aligned}$$

It is seen that

$$\hat{\boldsymbol{l}} = \cos\hat{\theta}\hat{\boldsymbol{e}}_1 + \sin\hat{\theta}\hat{\boldsymbol{e}}_2 \tag{624}$$

is the resulting line when the line $\hat{\boldsymbol{e}}_1$ rotated by an angle $\theta$ around $z$ and in addition translated a distance $d$ along the same axis. □

In the same way the line

$$\hat{\boldsymbol{k}} = \cos\hat{\theta}\hat{\boldsymbol{e}}_2 + \sin\hat{\theta}\hat{\boldsymbol{e}}_3 \tag{625}$$

is the line that results from rotation of the line $\hat{\boldsymbol{e}}_2$ by an angle $\theta$ around the $x$ axis and translating a distance $d$ along the same axis, while

$$\hat{\boldsymbol{n}} = \cos\hat{\theta}\hat{\boldsymbol{e}}_3 + \sin\hat{\theta}\hat{\boldsymbol{e}}_1 \tag{626}$$

is the line that results from rotation of the line $\hat{\boldsymbol{e}}_3$ by an angle $\theta$ around the $y$ axis and translating a distance $d$ along the same axis.

## 14 Velocity and angular velocity in terms of a twist

### 14.1 The twist corresponding to the derivative of a displacement

A twist is a screw which represents the velocity and angular velocity of a rigid body $B$ [11]. Let the motion of the rigid body be described by a homogeneous transformation matrix

$$\boldsymbol{T}_b^a = \begin{bmatrix} \boldsymbol{R}_b^a & \boldsymbol{p}_{ab}^a \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix} \tag{627}$$

from frame $a$ to frame $b$, where $a$ is a reference frame, and frame $b$ is fixed in the body $B$. Let the angular velocity from $a$ to $b$ be given i $a$ coordinates as $(\boldsymbol{\omega}_{ab}^a)^\times = \dot{\boldsymbol{R}}_b^a(\boldsymbol{R}_b^a)^{\mathrm{T}}$, and let the velocity of the origin of $b$ relative to the origin of $a$ be $\boldsymbol{v}_{ab}^a = \dot{\boldsymbol{p}}_{ab}^a$, so that

$$\dot{\boldsymbol{T}}_b^a = \begin{bmatrix} (\boldsymbol{\omega}_{ab}^a)^\times \boldsymbol{R}_b^a & \boldsymbol{v}_{ab}^a \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix} \tag{628}$$

This shows that the time derivative of the homogeneous transformation matrix $\boldsymbol{T}_b^a$ depends on the angular velocity $\boldsymbol{\omega}_{ab}^a$ and the velocity $\boldsymbol{v}_{ab}^a$. From the transformation rules $(\boldsymbol{\omega}_{ab}^a)^\times = \boldsymbol{R}_b^a(\boldsymbol{\omega}_{ab}^b)^\times \boldsymbol{R}^{\mathrm{T}}$ and $\boldsymbol{v}_{ab}^a = \boldsymbol{R}_b^a \boldsymbol{v}_{ab}^b$, it is seen that this can also be written in terms of the coordinates in the $b$ frame as

$$\dot{\boldsymbol{T}}_b^a = \begin{bmatrix} \boldsymbol{R}_b^a(\boldsymbol{\omega}_{ab}^b)^\times & \boldsymbol{R}_b^a \boldsymbol{v}_{ab}^b \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix} \tag{629}$$

The time derivative of the homogeneous transformation matrix can be written

$$\dot{\boldsymbol{T}}_b^a = \begin{bmatrix} \boldsymbol{R}_b^a & \boldsymbol{p}_{ab}^b \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} (\boldsymbol{\omega}_{ab}^b)^\times & \boldsymbol{v}_{ab}^b \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix} \tag{630}$$

The twist of the motion of frame $b$ relative to frame $a$ referenced to $b$ in the coordinates of $b$ is defined as the screw

$$\mathsf{t}_{ab/b}^b = \left\{ \begin{array}{c} \boldsymbol{\omega}_{ab}^b \\ \boldsymbol{v}_{ab}^b \end{array} \right\} \tag{631}$$

The twist $\mathsf{t}_{ab/b}^b$ has a matrix form

$$\bar{\mathsf{t}}_{ab/b}^b = \begin{bmatrix} (\boldsymbol{\omega}_{ab}^b)^\times & \boldsymbol{v}_{ab}^b \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix} \tag{632}$$

Then the time derivative of the homogeneous transformation matrix can be written

$$\dot{\boldsymbol{T}}_b^a = \boldsymbol{T}_b^a \bar{\mathsf{t}}_{ab/b}^b \tag{633}$$

which gives the following expression for the matrix form of the twist:

$$\bar{\mathsf{t}}_{ab/b}^b = (\boldsymbol{T}_b^a)^{-1} \dot{\boldsymbol{T}}_b^a \tag{634}$$

Note that this means that the twist $\mathsf{t}_{ab/b}^b$ can be defined from the homogeneous transformation matrix and its derivative according to this equation. This is a different approach from the geometric arguments presented above. In the following, it will be shown that also the screw properties of the twist follows from this definition.

### 14.2 Transformations of a twist

The derivative of the homogeneous transformation matrix can alternatively be developed with the velocity and the angular velocity in the $a$ fame. Then.

$$\dot{\boldsymbol{T}}_b^a = \begin{bmatrix} (\boldsymbol{\omega}_{ab}^a)^\times & (\boldsymbol{p}_{ab}^a)^\times \boldsymbol{\omega}_{ab}^a + \boldsymbol{v}_{ab}^a \\ \mathbf{0}^\mathrm{T} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{R}_b^a & \boldsymbol{p}_{ab}^a \\ \mathbf{0}^\mathrm{T} & 1 \end{bmatrix} \tag{635}$$

which can be written

$$\dot{\boldsymbol{T}}_b^a = \bar{\mathsf{t}}_{ab/a}^a \boldsymbol{T}_b^a \tag{636}$$

where

$$\bar{\mathsf{t}}_{ab/a}^a = \begin{bmatrix} (\boldsymbol{\omega}_{ab}^a)^\times & (\boldsymbol{p}_{ab}^a)^\times \boldsymbol{\omega}_{ab}^a + \boldsymbol{v}_{ab}^a \\ \mathbf{0}^\mathrm{T} & 0 \end{bmatrix} \tag{637}$$

is the matrix form of the twist

$$\mathsf{t}_{ab/a}^a = \begin{Bmatrix} \boldsymbol{\omega}_{ab}^a \\ (\boldsymbol{p}_{ab}^a)^\times \boldsymbol{\omega}_{ab}^a + \boldsymbol{v}_{ab}^a \end{Bmatrix} \tag{638}$$

which is the twist of the motion of frame $b$ relative to frame $a$ referenced to $a$ in the coordinates of $a$

It follows from $\dot{\boldsymbol{T}}_b^a = \bar{\mathsf{t}}_{ab/a}^a \boldsymbol{T}_b^a = \boldsymbol{T}_b^a \bar{\mathsf{t}}_{ab/b}^b$ that the matrix form of the twists transforms according to

$$\bar{\mathsf{t}}_{ab/a}^a = \boldsymbol{T}_b^a \bar{\mathsf{t}}_{ab/b}^b (\boldsymbol{T}_b^a)^{-1} \tag{639}$$

It is straightforward to verify that the usual transformation rule

$$\mathsf{t}_{ab/a}^a = \begin{bmatrix} \boldsymbol{R}_b^a & \mathbf{0} \\ (\boldsymbol{p}_{ab}^a)^\times \boldsymbol{R}_b^a & \boldsymbol{R}_b^a \end{bmatrix} \mathsf{t}_{ab/b}^b \tag{640}$$

for a screw is satisfied.

This means that $\mathsf{t}_{ab/a}^a$ is the screw that is obtained from $\mathsf{t}_{ab/b}^b$ when the reference point is moved by a distance $-\boldsymbol{p}_{ab}^a$ to the origin of frame $a$, and the coordinates are changed to the $a$ frame.

The twist $\mathsf{t}_{ab/a}^a$ is called the spatial velocity in [11], while the twist $\mathsf{t}_{ab/b}^b$ is called the body velocity. The velocity component $(\boldsymbol{p}_{ab}^a)^\times \boldsymbol{\omega}_{ab}^a + \boldsymbol{v}_{ab}^a$ in $\mathsf{t}_{ab/a}^a$ is the velocity of a point fixed in $B$ that passes through the origin of point $a$. It is not obvious that this is a useful physical velocity, but the definition of $\mathsf{t}_{ab/a}^a$ is important to establish the transformation rule for twists that describe the velocity and angular velocity between two frames.

It is noted that

$$\mathbf{Ad}_{\boldsymbol{T}_b^a} = \begin{bmatrix} \boldsymbol{R}_b^a & \mathbf{0} \\ (\boldsymbol{p}_{ab}^a)^\times \boldsymbol{R}_b^a & \boldsymbol{R}_b^a \end{bmatrix} \tag{641}$$

is called the adjoint transformation corresponding to $\boldsymbol{T}_b^a$. It is seen that the adjoint transformation is the same as the screw transformation.

### 14.3 Coordinate-free form of twists

A twist is a pair of vectors where one vector is an angular velocity, and the other vector is a velocity. To be more specific, in the displacement $\boldsymbol{T}_b^a$, the twist of a frame $b$ relative to a frame $a$ referenced to the origin of a frame $r$ is the collection of the angular velocity $\vec{\omega}_{ab}$ and the velocity $\vec{v}_{ab/r}$. The velocity $\vec{v}_{ab/r}$ is the velocity of a point fixed in frame $b$ that passes through the origin of frame $c$. In the case that $r$ is the frame $b$ this notation is simplifies as

$$\vec{v}_{ab/b} = \vec{v}_{ab} \tag{642}$$

A change of reference point is done with

$$\vec{v}_{ab/s} = \vec{v}_{ab/r} + \vec{\omega}_{ab} \times \vec{p}_{rs} \tag{643}$$

The coordinate-free description of the twist of $b$ relative to $a$ referenced to $r$ can then be written

$$\vec{t}_{ab/r} = (\vec{\omega}_{ab}, \vec{v}_{ab/r}) \tag{644}$$

The screw can be referenced to the origin of the frame $s$ by the transformation rule

$$\vec{t}_{ab/s} = (\vec{I}, \vec{p}_{sr}^{\times}) \cdot \vec{t}_{ab/r} \tag{645}$$

This gives

$$\vec{t}_{ab/s} = (\vec{I}, \vec{p}_{sr}^{\times}) \cdot (\vec{\omega}_{ab}, \vec{v}_{ab/s}) = (\vec{\omega}_{ab}, \vec{v}_{ab/r} + \vec{\omega}_{ab} \times \vec{p}_{rs}) \tag{646}$$

where $\vec{p}_{rs} = -\vec{p}_{sr}$ is the vector from $r$ to $s$, and $\vec{p}_{sr}^{\times} \cdot \vec{\omega}_{ab} = \vec{p}_{sr} \times \vec{\omega}_{ab} = \vec{\omega}_{ab} \times \vec{p}_{rs}$.

For the composite displacement $\boldsymbol{T}_c^a = \boldsymbol{T}_b^a \boldsymbol{T}_c^b$ the angular velocity will satisfy $\vec{\omega}_{ab} = \vec{\omega}_{ab} + \vec{\omega}_{bc}$, while the velocity will satisfy $\vec{v}_{ac/r} = \vec{v}_{ab/r} + \vec{v}_{bc/r}$. The twist of the displacement $\boldsymbol{T}_b^a$ referenced to $r$ is $\vec{t}_{ab/r}$, while the twist of $\boldsymbol{T}_c^b$ referenced to $r$ is $\vec{t}_{bc/r}$. Addition of the two twists gives

$$\vec{t}_{ab/r} + \vec{t}_{bc/r} = (\vec{\omega}_{ab} + \vec{\omega}_{bc}, \vec{v}_{ab/r} + \vec{v}_{bc/r}) = (\vec{\omega}_{ac}, \vec{v}_{ac/r}) \tag{647}$$

This means that the screw of the composite displacement is

$$\vec{t}_{ac/r} = \vec{t}_{ab/r} + \vec{t}_{bc/r} \tag{648}$$

This is easily extended to the composite displacements $\boldsymbol{T}_d^a = \boldsymbol{T}_b^a \boldsymbol{T}_c^b \boldsymbol{T}_d^c$ where the twist can be written

$$\vec{t}_{ad/r} = \vec{t}_{ab/r} + \vec{t}_{bc/r} + +\vec{t}_{cd/r} \tag{649}$$

Note that the twists of a composite displacement can only be added if they are referenced to the same point.

### 14.4 The instantaneous screw axis*

The instantaneous screw axis will be presented and discussed in this section. The instantaneous screw axis has the property that if a twist is referenced to any point on the instantaneous screw axis, then the velocity of the twist will be along the rotation axis. The instantaneous screw axis is used in the derivation of twists in [1].

Consider a rigid body $B$ and let the frame $b$ be fixed in $B$. The motion of the rigid body $B$ is described as a displacement $\boldsymbol{T}_b^a$ of the frame $b$ relative to frame $a$. The velocity $\vec{v}_{ab} = \vec{v}_{ab/b}$ of the origin of $b$ relative to the origin of $a$ can be described as $\vec{v}_{ab} = \vec{v}_{ab||} + \vec{v}_{ab\perp}$ where $\vec{v}_{ab||}$ is parallel to the angular velocity $\vec{\omega}_{ab}$, and $\vec{v}_{ab\perp}$ is perpendicular to the angular velocity.

There is a line where the points fixed in $B$ have velocity $\vec{v}_{||}$, that is, the velocity is only in the direction of the angular velocity. This line is called the instantaneous screw axis ISA [1]. Then the velocity of a point fixed in $B$ that passes through the ISA will have velocity

$$\vec{v}_{ab/\text{ISA}} = \vec{v}_{ab||} \tag{650}$$

A point $P$ fixed in $B$ that is not on the ISA, will have velocity given by $\vec{v}_{ab/P} = \vec{v}_{||} + \vec{\omega}_{ab} \times \vec{p}_{\text{ISA},P}$, where $\vec{p}_{\text{ISA},P}$ is the vector from ISA to $P$. In particular,

$$\vec{v}_{ab} = \vec{v}_{ab/b} = \vec{v}_{ab||} + \vec{\omega}_{ab} \times \vec{p}_{\text{ISA},b} \tag{651}$$

Then the twist of the displacement fram $a$ to $b$ can be referenced to a point on the $ISA$ as

$$\vec{t}_{ab/\text{ISA}} = (\vec{\omega}_{ab}, \vec{v}_{ab||}) \tag{652}$$

If the twist is referenced to the origin of $b$, it will be

$$\vec{t}_{ab/b} = (\vec{\omega}_{ab}, \vec{v}_{ab||} + \vec{\omega}_{ab} \times \vec{p}_{b,\text{ISA}}) = (\vec{\omega}_{ab}, \vec{v}_{ab/b}) \tag{653}$$

where the first component is the angular velocity of $b$ relative to $a$, and the second component is the velocity of the origin of frame $b$ relative to the origin of frame $a$.

**Example**

An interesting case appears if the instantaneous screw axis (ISA) of the displacement from $a$ to $b$ is a line through the origin of frame $a$ with direction vector $\vec{k}$ fixed in $a$. This will be the case if the rigid body is attached to the frame $a$ with a joint which has a rotation about the ISA, and a translation along ISA. Then $\vec{v}_{ab||} = v\vec{k}$ where $v$ is the magnitude of the velocity $\vec{v}_{ab||}$, and $\vec{\omega}_{ab} = \omega\vec{k}$ where $\omega$ is the magnitude of the angular velocity $\vec{\omega}_{ab}$. This gives

$$\vec{t}_{ab/b} = (\omega\vec{k}, v\vec{k} + \omega\vec{k} \times \vec{p}_{ab}) \tag{654}$$

### 14.5 The twist of a rotation about a fixed axis

To establish the required background to describe the twist of the end effector of a manipulator in terms of the twists of the joints, a simple mechanical system is studied where a link of length $h$ is rotated by an angle $\theta$ about a fixed axis $\mathsf{L}$. Let the frame $b$ be fixed in the link, and let the origin of frame $b$ be at the end of the link. Let frame $a$ be the fixed frame, and suppose that the axis of rotation is through the origin of frame $a$ in the direction of the $z$ axis of frame $a$. Then the twist axis of the joint referenced to frame $a$ and given in the coordinates of $a$ is

$$\mathsf{L} = \left\{ \begin{array}{c} \boldsymbol{z} \\ \boldsymbol{0} \end{array} \right\}, \quad \boldsymbol{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{655}$$

The vector from frame $a$ to frame $b$ is $\boldsymbol{p}_{ab}^b = [h, 0, 0]^{\mathrm{T}}$, which is fixed in frame $b$. The homogeneous transformation from frame $a$ to $b$ is

$$\boldsymbol{T}_b^a = \begin{bmatrix} \boldsymbol{R}_z(\theta) & \boldsymbol{R}_z(\theta)\boldsymbol{p}_{ab}^b \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix} \tag{656}$$

The twist of the joint is

$$\mathsf{t}_{ab/a}^a = \dot{\theta}\mathsf{L} = \left\{ \begin{array}{c} \dot{\theta}\boldsymbol{z} \\ \boldsymbol{0} \end{array} \right\} \tag{657}$$

Note that the twist is given in the coordinates of frame $a$, as indicated by the superscript, and it is referenced to the origin of frame $a$, as indicated by the subscript after the slash. The angular velocity part of the twist is $\dot{\theta}\boldsymbol{z}$, and the velocity is the velocity of the origin of frame $a$, which is zero.

This twist can be transformed to the origin of frame $b$ with the transformation

$$\mathsf{t}_{ab/b}^a = \begin{bmatrix} \boldsymbol{I} & 0 \\ (\boldsymbol{R}_b^a\boldsymbol{p}_{ba}^b)^\times & \boldsymbol{I} \end{bmatrix} \left\{ \begin{array}{c} \dot{\theta}\boldsymbol{z} \\ \boldsymbol{0} \end{array} \right\} = \left\{ \begin{array}{c} \dot{\theta}\boldsymbol{z} \\ \dot{\theta}\boldsymbol{z}^\times (\boldsymbol{R}_b^a\boldsymbol{p}_{ab}^b) \end{array} \right\} \tag{658}$$

Note that this twist is referenced to the origin of frame $b$, but it is given in the coordinates of frame $a$. Therefore, the screw transformation does not change the coordinates of the twist.

It is seen that if $\theta = 0$, then $\boldsymbol{R}_a^b = \boldsymbol{I}$, and the twist is

$$\mathsf{t}_{ab/b}^a = \left\{ \begin{array}{c} \dot{\theta}\boldsymbol{z} \\ \dot{\theta}h\boldsymbol{y} \end{array} \right\} \tag{659}$$

where $\boldsymbol{y} = [0, 1, 0]^{\mathrm{T}}$, which is in agreement with the velocity of the tip being $\dot{\theta}h$ in the $y$ direction.

### 14.6 The twists of a composite displacement

Consider the composite displacement

$$\boldsymbol{T}_c^a = \boldsymbol{T}_b^a\boldsymbol{T}_c^b = \begin{bmatrix} \boldsymbol{R}_c^a & \boldsymbol{p}_{ac}^a \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix} \tag{660}$$

where

$$\boldsymbol{T}_b^a = \begin{bmatrix} \boldsymbol{R}_b^a & \boldsymbol{p}_{ab}^a \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix}, \quad \boldsymbol{T}_c^b = \begin{bmatrix} \boldsymbol{R}_c^b & \boldsymbol{p}_{bc}^b \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix} \tag{661}$$

Define the twists

$$\bar{\mathsf{t}}_{ab/b}^b = (\boldsymbol{T}_b^a)^{-1}\dot{\boldsymbol{T}}_b^a, \quad \bar{\mathsf{t}}_{bc/c}^c = (\boldsymbol{T}_c^b)^{-1}\dot{\boldsymbol{T}}_c^b$$

with vector form

$$\mathsf{t}_{ab/b}^b = \left\{ \begin{array}{c} \boldsymbol{\omega}_{ab}^b \\ \boldsymbol{v}_{ab}^b \end{array} \right\}, \quad \mathsf{t}_{bc/c}^c = \left\{ \begin{array}{c} \boldsymbol{\omega}_{bc}^c \\ \boldsymbol{v}_{bc}^c \end{array} \right\}$$

Then the twist

$$\mathsf{t}_{ac/c}^c = \left\{ \begin{array}{c} \boldsymbol{\omega}_{ac}^c \\ \boldsymbol{v}_{ac}^c \end{array} \right\} \tag{662}$$

of the composite displacement is given in matrix form by

$$\bar{\mathsf{t}}^c_{ac/c} = (\boldsymbol{T}^a_c)^{-1}\dot{\boldsymbol{T}}^a_c \tag{663}$$

$$= (\boldsymbol{T}^b_c)^{-1}(\boldsymbol{T}^a_b)^{-1}(\dot{\boldsymbol{T}}^a_b\boldsymbol{T}^b_c + \boldsymbol{T}^a_b\dot{\boldsymbol{T}}^b_c) \tag{664}$$

$$= (\boldsymbol{T}^b_c)^{-1}(\boldsymbol{T}^a_b)^{-1}\dot{\boldsymbol{T}}^a_b\boldsymbol{T}^b_c + (\boldsymbol{T}^b_c)^{-1}\dot{\boldsymbol{T}}^b_c) \tag{665}$$

$$= \boldsymbol{T}^c_b\bar{\mathsf{t}}^b_{ab/b}(\boldsymbol{T}^c_b)^{-1} + \bar{\mathsf{t}}^c_{bc/c} \tag{666}$$

$$= \bar{\mathsf{t}}^c_{ab/c} + \bar{\mathsf{t}}^c_{bc/c} \tag{667}$$

It is seen that the twist of a composite displacement is the sum of the screws of the individual displacements, where all screws are referenced to the origin of the same reference frame, and all screws are given in the coordinates of the same coordinate frame, where the reference frame and the coordinate frame may be different. As an example, the twist of the composite displacement may be given in the coordinates of the $a$ frame as

$$\mathsf{t}^a_{ac/c} = \begin{bmatrix} \boldsymbol{R}^a_c & \mathbf{0} \\ \mathbf{0} & \boldsymbol{R}^a_c \end{bmatrix} \mathsf{t}^c_{ac/c} \tag{668}$$

where the vector form of the twist is used. This gives

$$\mathsf{t}^a_{ac/c} = \mathsf{t}^a_{ab/c} + \mathsf{t}^a_{bc/c} \tag{669}$$

It is straightforward to extend this to the composite displacement $\boldsymbol{T}^a_d = \boldsymbol{T}^a_b\boldsymbol{T}^b_c\boldsymbol{T}^c_d$ where the twist can be written

$$\mathsf{t}^a_{ad/d} = \mathsf{t}^a_{ab/d} + \mathsf{t}^a_{bc/d} + \mathsf{t}^a_{cd/d} \tag{670}$$

where the twists are referenced to the origin of the $d$ frame, while they are given in the coordinates of the $a$ frame.

**Example**

Consider again the displacement $\boldsymbol{T}^a_c = \boldsymbol{T}^a_b\boldsymbol{T}^b_c$. The twist of the composite displacement can be referenced to the $a$ frame through the transformation

$$\bar{\mathsf{t}}^a_{ac/a} = \boldsymbol{T}^a_c\bar{\mathsf{t}}^c_{ac/c}(\boldsymbol{T}^a_c)^{-1} = \boldsymbol{T}^a_c\left(\bar{\mathsf{t}}^c_{ab/c} + \bar{\mathsf{t}}^c_{bc/c}\right)(\boldsymbol{T}^a_c)^{-1} \tag{671}$$

Then it follows that

$$\mathsf{t}^a_{ac/a} = \mathsf{t}^a_{ab/a} + \mathsf{t}^a_{bc/a} \tag{672}$$

where all the twists are referenced to $a$. Alternatively, this could have been derived from

$$\bar{\mathsf{t}}^a_{ac/a} = \dot{\boldsymbol{T}}^a_c(\boldsymbol{T}^a_c)^{-1} \tag{673}$$

### 14.7 The link twists of a manipulator

Consider a rotary manipulator link described by Denavit-Hartenberg parameters. Suppose that this is link 1 of the manipulator, so that the transformation is from frame 0 til frame 1 according to the Denavit-Hartenberg convention. Moreover, suppose that frame 0 is fixed so that the velocity of the origin of frame 0 is zero. Then, according to the Denavit-Hartenberg convention the joint axis is the $z$ axis of frame 0, which has the coordinate vector $\boldsymbol{z} = [0,0,1]^{\mathrm{T}}$ in frame 0. The joint axis is through the origin of frame 0, which means that the joint axis is the line

$$\mathsf{L}_{1/0}^{0} = \left\{ \begin{array}{c} \boldsymbol{z} \\ \mathbf{0} \end{array} \right\} \tag{674}$$

which is given in the coordinates of 0 as a screw referenced to the origin of frame 0.

For a rotary joint the twist from frame 0 to frame 1 is then

$$\mathsf{t}_{01/0}^{0} = \left\{ \begin{array}{c} \boldsymbol{\omega}_{01}^{0} \\ \mathbf{0} \end{array} \right\} = \dot{\theta}_1 \left\{ \begin{array}{c} \boldsymbol{z} \\ \mathbf{0} \end{array} \right\} = \dot{\theta}_1 \mathsf{L}_{1/0}^{0} \tag{675}$$

where the twist is referenced to frame 0. Note that the components of the twist are the angular velocity of frame 1 relative to frame 0, and the velocity of the point in link 1 that is incident with the origin in frame 0. Since link 1 is attached to the link 0 in joint 1, this velocity is zero, else the joint would break apart.

For a prismatic joint 1 the corresponding twist is

$$\mathsf{t}_{0,1/0}^{0} = \left\{ \begin{array}{c} \mathbf{0} \\ \dot{d}_i \boldsymbol{z} \end{array} \right\} = \dot{d}_i \mathsf{M}_{1/0}^{0} \tag{676}$$

where the screw axis

$$\mathsf{M}_{1/0}^{0} = \left\{ \begin{array}{c} \mathbf{0} \\ \boldsymbol{z} \end{array} \right\} \tag{677}$$

is a line at infinity.

For the rotary joint $i$, $i = 1, \ldots, 6$ the twist from frame $i-1$ to $i$ referenced to the origin of frame $i-1$ is found in the same way to be

$$\mathsf{t}_{i-1,i/i-1}^{i-1} = \left\{ \begin{array}{c} \dot{\theta}_1 \boldsymbol{z} \\ \mathbf{0} \end{array} \right\} = \dot{\theta}_i \mathsf{L}_{i/i-1}^{i-1}, \quad \mathsf{L}_{i/i-1}^{i-1} = \left\{ \begin{array}{c} \boldsymbol{z} \\ \mathbf{0} \end{array} \right\} \tag{678}$$

where the axis of the screw is the line $\mathsf{L}_{i/i-1}^{i-1}$.

This twist is given in the coordinates of 0 and is referenced to frame $e$ with the transformation

$$\mathsf{t}_{i-1,i/e}^{0} = \left[ \begin{array}{cc} \boldsymbol{R}_{i-1}^{0} & 0 \\ (\boldsymbol{p}_{e,i-1}^{0})^{\times} \boldsymbol{R}_{i-1}^{0} & \boldsymbol{R}_{i-1}^{0} \end{array} \right] \dot{\theta}_i \mathsf{L}_{i/i-1}^{i-1} = \dot{\theta}_i \mathsf{L}_{i/e}^{0} \tag{679}$$

The axis of the twist $\mathsf{t}_{i-1,i/e}^{0}$ is the line

$$\mathsf{L}_{i/e}^{0} = \left\{ \begin{array}{c} \boldsymbol{z}_{i-1}^{0} \\ (\boldsymbol{p}_{e,i-1}^{0})^{\times} \boldsymbol{z}_{i-1}^{0} \end{array} \right\} \tag{680}$$

where $\boldsymbol{z}_{i-1}^{0}$ is the unit vector of the $z$ axis of frame $i-1$ in the coordinates of frame 0. $\boldsymbol{z}_{i-1}^{0} = \boldsymbol{R}_{i-1}^{0} \boldsymbol{z}$ is computed as the last column of $\boldsymbol{R}_{i-1}^{0}$.

For the prismatic joint $i$, $i = 1, \ldots, 6$ the twist from frame $i-1$ to $i$ is found to be

$$\mathsf{t}_{i-1,i/i}^{i-1} = \left\{ \begin{array}{c} \mathbf{0} \\ \dot{d}_i \boldsymbol{z} \end{array} \right\} = \dot{d}_i \mathsf{M}_{i/i-1}^{i-1}, \quad \mathsf{M}_{i/i-1}^{i-1} = \left\{ \begin{array}{c} \mathbf{0} \\ \dot{d}_i \boldsymbol{z} \end{array} \right\} \tag{681}$$

where the twist is referenced to the origin of frame $i-1$. This twist is referenced to frame $e$ with the transformation

$$\mathsf{t}^0_{i-1,i/e} = \begin{bmatrix} \boldsymbol{R}^0_{i-1} & 0 \\ (\boldsymbol{p}^0_{e,i-1})^\times \boldsymbol{R}^0_{i-1} & \boldsymbol{R}^0_{i-1} \end{bmatrix} \dot{d}_i \mathsf{M}^{i-1}_{i/i-1} = \dot{d}_i \dot{d}_i \mathsf{M}^0_{i/e} \tag{682}$$

where the axis of the twist is the line at infinity given by

$$\mathsf{M}^0_{i/e} = \left\{ \begin{array}{c} \mathbf{0} \\ \boldsymbol{z}^0_{i-1} \end{array} \right\} \tag{683}$$

### 14.8 Link twists and the Jacobian for a manipulator

Consider a manipulator with 6 links where the links are assigned according to the Denavit-Hartenberg convention. The displacement of the end effector relative to the base frame 0 is given by $\boldsymbol{T}^0_e = \boldsymbol{T}^0_1 \boldsymbol{T}^1_2 \ldots \boldsymbol{T}^5_e$. The twist of the composite displacement $\boldsymbol{T}^0_e$ is written

$$\mathsf{t}^0_{0e/e} = \left\{ \begin{array}{c} \boldsymbol{\omega}^0_{0e} \\ \boldsymbol{v}^0_{0e} \end{array} \right\} \tag{684}$$

where the twist is referenced to the end effector frame given in the coordinates of frame 0. This twist can then be written as the sum of the joint twists $\mathsf{t}^0_{i-1,i/e}$ as

$$\mathsf{t}^0_{0e/e} = \mathsf{t}^0_{0,1/e} + \mathsf{t}^0_{1,2/e} + \ldots + \mathsf{t}^0_{5,e/e} \tag{685}$$

Define the generalized coordinate of joint $i$ by

$$q_i = \left\{ \begin{array}{ll} \theta_i, & \text{joint } i \text{ is rotary} \\ d_i, & \text{joint } i \text{ is prismatic} \end{array} \right.$$

and define

$$\mathsf{J}_i = \left\{ \begin{array}{ll} \mathsf{L}^0_{i/e}, & \text{joint } i \text{ is rotary} \\ \mathsf{M}^0_{i/e}, & \text{joint } i \text{ is prismatic} \end{array} \right.$$

Then

$$\mathsf{t}^0_{0e/e} = \dot{q}_1 \mathsf{J}_1 + \dot{q}_2 \mathsf{J}_2 + \ldots + \dot{q}_6 \mathsf{J}_6 = \sum_{i=1}^{6} \dot{q}_i \mathsf{J}_i \tag{686}$$

In coordinate form this is written

$$\begin{bmatrix} \boldsymbol{\omega}^0_{0e} \\ \boldsymbol{v}^0_{0e} \end{bmatrix} = \boldsymbol{J} \dot{\boldsymbol{q}} \tag{687}$$

where the columns of $\boldsymbol{J}$ are the coordinate vectors of the screw vectors $\mathsf{J}_i$. This is the familiar equation for the geometric Jacobian, except that the twist is then usually written $[(\boldsymbol{v}^0_{0e})^{\mathrm{T}}, (\boldsymbol{\omega}^0_{0e})^{\mathrm{T}}]^{\mathrm{T}}$.

### 14.9 Link twists of a vehicle-manipulator system

In this section the dynamics of a vehicle-manipulator system is described in terms of twists and wrenches [1, 11, 9]. In this problem the base of the manipulator is a moving vehicle. Let $n$ be an inertial frame, while 0 is the base frame of the manipulator, and the frames 1 - 6 are the link frames of the manipulator links according to the Denavit-Hartenberg convention. The base is supposed to be a moving vehicle.

In the equations of motion there is a need to express the link twists in the form

$$\mathsf{t}^i_{ni/i} = \left\{ \begin{array}{c} \boldsymbol{\omega}^i_{ni} \\ \boldsymbol{v}^i_{ni/i} \end{array} \right\}, \quad i = 0, 1, \ldots, 6 \tag{688}$$

where the twist is referenced to the origin of $i$ if link $i$. This point is fixed in frame $i$.

The twist of link $i$ can be written

$$\mathsf{t}^i_{ni/i} = \mathsf{t}^i_{n0/i} + \mathsf{t}^i_{01/i} + \ldots + \mathsf{t}^i_{i-1,i/i} \tag{689}$$

which is the sum of the twist of the base, and the relative twists of the links. The twist of the base is supposed to be

$$\mathsf{t}^0_{n0/0} = \left\{ \begin{array}{c} \boldsymbol{\omega}^0_{n0} \\ \boldsymbol{v}^0_{n0} \end{array} \right\} \tag{690}$$

The joint links are supposed to be rotary, and the relative twists of the links are

$$\mathsf{t}^{j-1}_{j-1,j/j-1} = \dot{\theta}_j \mathsf{L}^{j-1}_{j/j-1}, \quad \mathsf{L}^{j-1}_{j/j-1} = \left\{ \begin{array}{c} \boldsymbol{z} \\ \boldsymbol{0} \end{array} \right\} \tag{691}$$

where the axis of the twist is a line through the origin of frame $j-1$ in the direction of the $z$ axis of frame $j-1$.

The twist of link $i$ is

$$\mathsf{t}^i_{ni/i} = \mathsf{t}^i_{n0/i} + \mathsf{t}^i_{01/i} + \ldots + \mathsf{t}^i_{i-1,i/i} = \mathsf{t}^i_{n0/i} + \sum_{j=1}^{i-1} \dot{\theta}_j \mathsf{L}^i_{j/i} \tag{692}$$

Let the generalized velocities of the vehicle-manipulator system be given by

$$\boldsymbol{u} = [(\boldsymbol{\omega}^0_{n0})^{\mathrm{T}}, (\boldsymbol{v}^0_{n0})^{\mathrm{T}}, \dot{\theta}_1, \ldots, \dot{\theta}_6]^{\mathrm{T}} \tag{693}$$

Then

$$\mathsf{t}^i_{ni/i} = \boldsymbol{P}_i \boldsymbol{u} \tag{694}$$

where $\boldsymbol{P}_i$ is the Jacobian of link $i$, which is given by

$$\boldsymbol{P}_i = \left[ \begin{array}{cccccc} \hat{\boldsymbol{T}}^i_0 & \mathsf{L}^i_{1/i} & \ldots & \mathsf{L}^i_{i/i} & \boldsymbol{0} & \ldots & \boldsymbol{0} \end{array} \right] \tag{695}$$

Here

$$\hat{\boldsymbol{T}}^i_0 = \left[ \begin{array}{cc} \boldsymbol{R}^i_0 & \boldsymbol{0} \\ (\boldsymbol{p}^i_{i0})^\times \boldsymbol{R}^i_0 & \boldsymbol{R}^i_0 \end{array} \right] \tag{696}$$

and

$$\mathsf{L}^i_{j/i} = \left[ \begin{array}{cc} \boldsymbol{R}^i_{j-1} & \boldsymbol{0} \\ (\boldsymbol{p}^i_{i,j-1})^\times \boldsymbol{R}^i_{j-1} & \boldsymbol{R}^i_{j-1} \end{array} \right] \mathsf{L}^{j-1}_{j/j-1} = \left\{ \begin{array}{c} \boldsymbol{z}^i_{j-1} \\ (\boldsymbol{p}^i_{i,j-1})^\times \boldsymbol{z}^i_{j-1} \end{array} \right\}, \quad j = 1, \ldots, i \tag{697}$$

## 15 Forces and torques on a rigid body in terms of a wrench

### 15.1 The wrench as a screw representation of forces and moments

A wrench is a screw description of the forces and moments acting on a rigid body. Before introducing the wrench a discussion on the description of equivalent descriptions of forces and moments will be presented. In this discussion a force vector is given in a coordinate-free description $\vec{F}$, a moment is given by $\vec{N}$. The moment of the force $\vec{F}$ about appoint $P$ is given by $\vec{N} = \vec{r} \times \vec{F}$ where $\vec{r}$ is the position vector from the point $P$ to the line of action of the force.

### 15.2 Forces and torques acting on a rigid body

Consider a rigid body $B$ and a set $S$ of $n_F$ forces $\vec{F}_i$, $i = 1, \ldots, n_F$, acting on the body $B$. Each force has a line of action, which is a line in the direction of the force through the point where the force is acting on the body.

The combined action of the set $S$ of forces can be described in terms of a resultant force $\vec{F}_S^{(r)}$ and the moment $\vec{N}_{S/P}$ on $B$ of the set $S$ about a point $P$. The resultant force $\vec{F}_S^{(r)}$ of the set $S$ of forces is the vector

$$\vec{F}_S^{(r)} = \sum_{i=1}^{n_F} \vec{F}_i \tag{698}$$

while the moment on $B$ of the set $S$ about a point $P$ is

$$\vec{N}_{S/P} = \sum_{i=1}^{n_F} \vec{r}_{Pi} \vec{F}_i \tag{699}$$

where $\vec{r}_{Pi}$ is the position vector from the point $P$ to a point on the line of action of force $\vec{F}_i$. This moment may include a torque, which is represented by a force couple. It is interesting to note that the resultant force is a sum of forces, where each force has a line of action. In contrast to this, the resultant force cannot be considered to have a line of action.

The moment about some other point $Q$ is given by

$$\vec{N}_{S/Q} = \sum_{i=1}^{n_F} \vec{r}_{Qi} \vec{F}_i = \sum_{i=1}^{n_F} \vec{r}_{Pi} \vec{F}_i + \sum_{i=1}^{n_F} \vec{r}_{QP} \vec{F}_i \tag{700}$$

which gives the following expression for the moment about the point $Q$.

$$\vec{N}_{S/Q} = \vec{N}_{S/P} + \vec{r}_{QP} \times \vec{F}_S^{(r)} \tag{701}$$

This shows that the moment about $Q$ is the moment about $P$ plus the moment that would result from the resultant force if the resultant force had line of action through the point $P$. This description has the problem that the resultant force does not have a well-defined line of action. Therefore it is common practice [7] to represent the set $S$ of forces by an equivalent set $\Sigma_p$ with a force $\vec{F}_{\Sigma_p}$ and torque $\vec{T}_{\Sigma_p}$ where

$$\vec{F}_{\Sigma_p} = \vec{F}_S^{(r)}, \quad \vec{T}_{\Sigma_p} = \vec{N}_{S/P} \tag{702}$$

where the force $\vec{F}_{\Sigma_p}$ is assigned a line of action through the point $P$. Then, the moment about a point $Q$ will be

$$\vec{N}_{\Sigma_p/Q} = \vec{T}_{\Sigma_p} + \vec{r}_{QP} \times \vec{F}_{\Sigma_p} \tag{703}$$

which shows that the sets $S$ and $\Sigma_p$ are equivalent.

It is noted that an alternative equivalent set $\Sigma_q$ can be defined with a force $\vec{F}_{\Sigma_q} = \vec{F}_{\Sigma_p}$ with line of action through the point $Q$, and a torque $\vec{T}_{\Sigma_q} = \vec{T}_{\Sigma_p} + \vec{r}_{QP} \times \vec{F}_{\Sigma_p}$.

### 15.3 The wrench as a screw representation of forces and moments

A frame $b$ is fixed in the rigid body $B$, and the set $\Sigma_p$ with a force $\vec{F}_{\Sigma_p}$ and torque $\vec{T}_{\Sigma_p}$ is given in the coordinates of frame $b$ as $\boldsymbol{F}^b_{\Sigma_p}$ and $\boldsymbol{T}^b_{\Sigma_p}$. Consider the following screw

$$\boldsymbol{w}^b_{\Sigma_p/P} = \left\{ \begin{array}{c} \boldsymbol{F}^b_{\Sigma_p} \\ \boldsymbol{T}^b_{\Sigma_p} \end{array} \right\} \tag{704}$$

This is the wrench of $\Sigma_p$ with reference to the point $P$ in the coordinates of $b$. The wrench of a set $\Sigma_p$ with reference point $P$ is a screw where the direction vector is the force of $\Sigma_p$, and the moment is the torque of $\Sigma_p$. Note that the force must have a line of action in this description.

This wrench of $\Sigma_p$ with reference point $P$ can be transformed to a reference point $Q$ with the usual screw transformation

$$\boldsymbol{w}^b_{\Sigma_p/Q} = \left[ \begin{array}{cc} \boldsymbol{I} & \boldsymbol{0} \\ (\boldsymbol{r}^b_{QP})\times & \boldsymbol{I} \end{array} \right] \boldsymbol{w}^b_{\Sigma_p/P} \tag{705}$$

which gives

$$\left\{ \begin{array}{c} \boldsymbol{F}^b_{\Sigma_q} \\ \boldsymbol{T}^b_{\Sigma_q} \end{array} \right\} = \left[ \begin{array}{cc} \boldsymbol{I} & \boldsymbol{0} \\ (\boldsymbol{r}^b_{QP})\times & \boldsymbol{I} \end{array} \right] \left\{ \begin{array}{c} \boldsymbol{F}^b_{\Sigma_q} \\ \boldsymbol{T}^b_{\Sigma_q} \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{F}^b_{\Sigma_q} \\ \boldsymbol{T}^b_{\Sigma_q} + (\boldsymbol{r}^b_{QP}) \times \boldsymbol{F}^b_{\Sigma_q} \end{array} \right\} \tag{706}$$

It is seen that the resulting wrench is a screw where the direction vector is the force $\boldsymbol{F}^b_{\Sigma_q}$ with line of action through $P$, and the moment $\boldsymbol{T}^b_{\Sigma_q} + (\boldsymbol{r}^b_{QP}) \times \boldsymbol{F}^b_{\Sigma_q}$ of $\Sigma_p$ about $Q$. This means that the wrench maintains its physical interpretation under screw transformations.

### 15.4 The wrench axis

Consider a rigid body $B$ where the combined action of the forces acting on the body is represented by set $\Sigma_p$ with a force $\vec{F}_{\Sigma_p}$ with line of action through a point $P$ that is fixed in $B$, and a torque $\vec{T}_{\Sigma_p}$. The torque is written

$$\vec{T}_{\Sigma_p} = \vec{T}_{||} + \vec{T}_{\perp} \tag{707}$$

where $\vec{T}_{||}$ is parallel to the line of action of the force $\vec{F}_{\Sigma_p}$, and $\vec{T}_{\perp}$ is perpendicular to the force $\vec{F}$. The moment about a point $Q$ is

$$\vec{N}_{\Sigma_p/Q} = \vec{T}_{\Sigma_p} + \vec{r}_{QP} \times \vec{F}_{\Sigma_p} = \vec{T}_{||} + \vec{T}_{\perp} + \vec{r}_{QP} \times \vec{F}_{\Sigma_p} \tag{708}$$

Her the moment $\vec{r}_{QP} \times \vec{F}_{\Sigma_p}$ is perpendicular to the line of action of the force. This means that whenever the force is nonzero, it is possible to select the point $Q$ so that $\vec{T}_{\perp} = -\vec{r}_{QP} \times \vec{F}_{\Sigma_p}$. For this particular point $Q$, the moment will be $\vec{N}_{\Sigma_p/Q} = \vec{T}_{||}$, which is a torque about the line of action of the force.

Then the combined action of the forces acting in $B$ can be represented by the equivalent set $\Sigma_q$ defined by the force $\vec{F}_{\Sigma_q}$ with line of action through the point $Q$, and a torque $\vec{T}_{\Sigma_q} = \vec{T}_{||}$ along the line of axis of the force $\vec{F}_{\Sigma_q}$. The line of action of the force is called the wrench axis WA. The wrench referenced to any point on the wrench axis is

$$\boldsymbol{w}^b_{\Sigma_q/\mathrm{WA}} = \left\{ \begin{array}{c} \boldsymbol{F}^b_{\Sigma_q} \\ \boldsymbol{T}^b_{\Sigma_q} \end{array} \right\} \tag{709}$$

where the force and the torque vectors are parallel.

### 15.5 Systems of wrenches

The concept of a wrench axis is developed in this section. The wrench axis has the property that if a wrench is referenced to any point on the wrench axis, then the moment of the wrench will be along the line of action of the force. This is mainly interesting from a theoretical point of view, and is included to give more insight into the physics of the problem. Consider a set of wrenches $\boldsymbol{w}^{b_i}_{i/c_i}$ that are given in the coordinates of frame $b_i$, and has the origin of frame $c_i$ as a reference point. This wrench can be transformed to the coordinates of frame $b$ and to the origin of frame $c$ as the reference point through the screw transformation

$$\boldsymbol{w}^b_{i/c} = \left[ \begin{array}{cc} \boldsymbol{R}^b_{b_i} & \boldsymbol{0} \\ (\boldsymbol{p}^b_{c,c_i})^{\times} \boldsymbol{R}^b_{b_i} & \boldsymbol{R}^b_{b_i} \end{array} \right] \boldsymbol{w}^{b_i}_{i/c_i} \tag{710}$$

In the special case where coordinate frames $b$ and $b_i$ coincide with the reference frames $c$ and $c_i$ this transformation becomes

$$\boldsymbol{w}^b_{i/c} = \mathbf{Ad}_{\boldsymbol{T}^b_{b_i}} \boldsymbol{w}^{b_i}_{i/b_i} \tag{711}$$

where

$$\mathbf{Ad}_{\boldsymbol{T}^b_{b_i}} = \left[ \begin{array}{cc} \boldsymbol{R}^b_{b_i} & \boldsymbol{0} \\ (\boldsymbol{p}^b_{b,b_i})^{\times} \boldsymbol{R}^b_{b_i} & \boldsymbol{R}^b_{b_i} \end{array} \right] \tag{712}$$

is the adjoint transformation of

$$\boldsymbol{T}^b_{b_i} = \left[ \begin{array}{cc} \boldsymbol{R}^b_{b_i} & \boldsymbol{p}^b_{b,b_i} \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{array} \right] \tag{713}$$

Then a set of $n_w$ wrenches can be transformed to the coordinates of $b$ and the reference point in the origin of $b$ with the transformation

$$\boldsymbol{w}^b_{i/c} = \sum_{i=1}^{n_w} \mathbf{Ad}_{\boldsymbol{T}^b_{b_i}} \boldsymbol{w}^{b_i}_{i/b_i} \tag{714}$$

### 15.6 Reciprocity of twist and wrench

Consider a twist

$$\boldsymbol{t} = \left\{ \begin{array}{c} \boldsymbol{\omega} \\ \boldsymbol{v} \end{array} \right\} \tag{715}$$

and a wrench

$$\boldsymbol{w} = \left\{ \begin{array}{c} \boldsymbol{F} \\ \boldsymbol{T} \end{array} \right\} \tag{716}$$

that are given in the coordinates of the same frame, and with the same reference point. Then the inner product of the twist and the wrench is

$$\boldsymbol{t} \cdot \boldsymbol{w} = \left\{ \begin{array}{c} \boldsymbol{\omega} \cdot \boldsymbol{F} \\ \boldsymbol{v} \cdot \boldsymbol{F} + \boldsymbol{\omega} \cdot \boldsymbol{T} \end{array} \right\} \tag{717}$$

The first term of the inner product has no physical significance. The twist and the wrench are said to be reciprocal if the second term of the inner product is zero, that is,

$$\boldsymbol{v} \cdot \boldsymbol{F} + \boldsymbol{\omega} \cdot \boldsymbol{T} = 0 \tag{718}$$

It is seen that if the twist and the wrench are reciprocal, then the work done on the rigid body by the wrench is zero.

### 15.7 Invariance of reciprocity under displacements

Consider the twist $\hat{\boldsymbol{t}} = \boldsymbol{\omega} + \varepsilon \boldsymbol{v}$ and the wrench $\hat{\boldsymbol{w}} = \boldsymbol{F} + \varepsilon \boldsymbol{T}$. In the dual number formulation the screw transformation is represented by the dual matrix $\hat{\boldsymbol{A}} = \boldsymbol{R} + \varepsilon(\boldsymbol{d}^{\times}\boldsymbol{R} + \boldsymbol{R})$. The transformed screws are

$$\hat{\boldsymbol{t}}' = \hat{\boldsymbol{A}}\hat{\boldsymbol{t}} = (\boldsymbol{R} + \varepsilon(\boldsymbol{d}^{\times}\boldsymbol{R} + \boldsymbol{R}))(\boldsymbol{\omega} + \varepsilon\boldsymbol{v}) = \boldsymbol{R}\boldsymbol{\omega} + \varepsilon(\boldsymbol{R}\boldsymbol{v} + \boldsymbol{d}^{\times}\boldsymbol{R}\boldsymbol{\omega}) \tag{719}$$

$$\hat{\boldsymbol{w}}' = \hat{\boldsymbol{A}}\hat{\boldsymbol{w}} = (\boldsymbol{R} + \varepsilon(\boldsymbol{d}^{\times}\boldsymbol{R} + \boldsymbol{R}))(\boldsymbol{F} + \varepsilon\boldsymbol{T}) = \boldsymbol{R}\boldsymbol{F} + \varepsilon(\boldsymbol{R}\boldsymbol{T} + \boldsymbol{d}^{\times}\boldsymbol{R}\boldsymbol{F}) \tag{720}$$

The inner product of the transformed screws is

$$\hat{\boldsymbol{t}}' \cdot \hat{\boldsymbol{w}}' = [\boldsymbol{R}\boldsymbol{\omega} + \varepsilon(\boldsymbol{R}\boldsymbol{v} + \boldsymbol{d}^{\times}\boldsymbol{R}\boldsymbol{\omega})] \cdot [\boldsymbol{R}\boldsymbol{F} + \varepsilon(\boldsymbol{R}\boldsymbol{T} + \boldsymbol{d}^{\times}\boldsymbol{R}\boldsymbol{F})] \tag{721}$$

$$= \boldsymbol{R}\boldsymbol{\omega} \cdot \boldsymbol{R}\boldsymbol{F} + \varepsilon(\boldsymbol{R}\boldsymbol{v} \cdot \boldsymbol{R}\boldsymbol{F} + \boldsymbol{R}\boldsymbol{\omega} \cdot \boldsymbol{R}\boldsymbol{T} + \boldsymbol{d}^{\times}\boldsymbol{R}\boldsymbol{\omega} \cdot \boldsymbol{R}\boldsymbol{F} + \boldsymbol{R}\boldsymbol{\omega} \cdot \boldsymbol{d}^{\times}\boldsymbol{R}\boldsymbol{F}) \tag{722}$$

$$= \boldsymbol{\omega}^{\mathrm{T}}\boldsymbol{R}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{F} + \varepsilon(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{R}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{F} + \boldsymbol{\omega}^{\mathrm{T}}\boldsymbol{R}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{T} + \boldsymbol{\omega}^{\mathrm{T}}\boldsymbol{R}^{\mathrm{T}}(\boldsymbol{d}^{\times})^{\mathrm{T}}\boldsymbol{R}\boldsymbol{F} + \boldsymbol{\omega}^{\mathrm{T}}\boldsymbol{R}^{\mathrm{T}}\boldsymbol{d}^{\times}\boldsymbol{R}\boldsymbol{F}) \tag{723}$$

$$= \boldsymbol{\omega} \cdot \boldsymbol{F} + \varepsilon(\boldsymbol{v} \cdot \boldsymbol{F} + \boldsymbol{\omega} \cdot \boldsymbol{T}) \tag{724}$$

$$= \hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{w}} \tag{725}$$

where it is used that the skew symmetric form satisfies $\boldsymbol{d}^{\times} + (\boldsymbol{d}^{\times})^{\mathrm{T}} = 0$. It follows that the scalar product of a twist and a wrench is invariant under displacements, and that reciprocity of a twist and a wrench is still satisfied when the twist and the wrench undergo the same displacement.

## 16 Parallel manipulators

### 16.1 Introduction

The usual industrial manipulators are of the serial-link type, where each link of the robot is connected with a joint with one degree of freedom. An $n$ link manipulator will have $\boldsymbol{n}$ links connected with $n$ joints, and there are $n$ independent joint variables $q_i$, $i = 1, \ldots, n$.

Another type of manipulator is the parallel manipulator [10]. This is a type of manipulator where there is typically a moving platform with $m$ degrees of freedom that is connected to a fixed base by $m$ legs or limbs. The design of the parallel manipulator is said to be symmetrical if the limbs have the same mechanical design, and each limb has one degree of freedom controlled by an actuator [19]. The forward kinematic problem will then be to determine the position and orientation of the platform for when the limb variables are given. The inverse kinematic problem is to determine the limb variables when the position and orientation of the platform is given.

### 16.2 The Stewart platform

A well-known parallel manipulator is the Stewart platform [18]. This manipulator has a controlled platform with 6 degrees of freedom, and is controlled with 6 limbs, where each limb is a prismatic joint where the length of the limb is controlled with an actuator. Each limb is connected with a spherical joint to the base at one end, and is connected with a spherical joint to the controlled platform at the other end. The limb has a prismatic joint. Therefore, this arrangement is called a 6SPS design, which is a design with 6 limbs, and each limb is a spherical-prismatic-spherical arrangement.

### 16.3 Inverse kinematics

A frame $a$ is fixed to the base, and a frame $b$ is fixed to the moving platform. The displacement from frame $a$ to frame $b$ is given by the homogeneous transformation matrix

$$\boldsymbol{T}_b^a = \begin{bmatrix} \boldsymbol{R}_b^a & \boldsymbol{p} \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix} \tag{726}$$

Limb $i$ is fixed with a spherical joint in the base at the position $\boldsymbol{a}_i$ relative to frame $a$ in the coordinates of $a$, and with a spherical joint in the platform at position $\boldsymbol{b}_i$ relative to the $b$ frame in the coordinates of $b$. The vectors $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$ are constant design parameters of the manipulator. The length of limb $i$ is given by the length $d_i$ of the vector $\boldsymbol{d}_i$ from the spherical joint in the base to the spherical joint at the platform. Then $\boldsymbol{p} = \boldsymbol{a}_i + \boldsymbol{d}_i - \boldsymbol{R}_b^a \boldsymbol{b}_i$, and

$$\boldsymbol{d}_i = \boldsymbol{p} - \boldsymbol{a}_i + \boldsymbol{R}_b^a \boldsymbol{b}_i \tag{727}$$

and the inverse kinematics is given by joint lengths

$$d_i = \sqrt{\boldsymbol{d}_i^{\mathrm{T}} \boldsymbol{d}_i}, \quad i = 1, \ldots, 6 \tag{728}$$

where $\boldsymbol{d}_i$ is calculated from (727) when $\boldsymbol{T}_b^a$ is given. The joint lengths must be positive, so the positive solution of $d_i^2 = \boldsymbol{d}_i^{\mathrm{T}} \boldsymbol{d}_i$ can always be used.

### 16.4 Jacobians

The Jacobian for a parallel manipulator gives a linear mapping between the joint velocities and the linear and angular velocity of the platform. This mapping can be written

$$\boldsymbol{J}_w \boldsymbol{w} = \boldsymbol{J}_q \dot{\boldsymbol{q}} \tag{729}$$

where $\boldsymbol{w} = [\boldsymbol{v}^{\mathrm{T}}, \boldsymbol{\omega}^{\mathrm{T}}]^{\mathrm{T}}$ is the twist vector of the platform, where the difference compared to a serial-link manipulator is that there is a matrix $\boldsymbol{J}_w$ in the expression. The Jacobian $\boldsymbol{J}$ of the parallel manipulator is then defined by

$$\dot{\boldsymbol{q}} = \boldsymbol{J}\boldsymbol{w}, \quad \boldsymbol{J} = \boldsymbol{J}_q^{-1}\boldsymbol{J}_w \tag{730}$$

It is noted that the Jacobian of a parallel manipulator corresponds to the inverse Jacobian of a serial-link manipulator.

### 16.5 Singularities

The singularities of a parallel manipulator are the singularities of $\boldsymbol{J}_q$, which are called inverse kinematic singularities, and the singularities of $\boldsymbol{J}_w$, which are called the direct kinematic singularities. Singularities where both $\boldsymbol{J}_q$ and $\boldsymbol{J}_w$ are singular are called combined singularities.

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